MATH 54 - WORKSHEET ~~13~~ 14
GENERALIZED FOURIER ~~SEIRES~~ SERIES

We defined Fourier series by way of orthogonal projection, using the inner product

$$\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)\,dx$$

on the space $C[-L, L]$ and the orthogonal set of functions

$$\{1, \sin(\pi x/L), \cos(\pi x/L), \sin(2\pi x/L), \cos(2\pi x/L), \dots\}.$$

This inner product is very natural, and this orthogonal set of functions is very convenient, but there are many other inner products on $C[-L, L]$ and many other orthogonal sets of functions. Orthogonal projection for a different inner product and/or a different orthogonal set gives a *generalized Fourier series.* In this worksheet, we will examine a specific example: Chebyshev series.

(1) First, we examine other inner products. Show that if $w(x)$ is any continuous function on $[-L, L]$ which is positive everywhere (except possibly at $\pm L$), then $\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)w(x)\,dx$ is an inner product (check symmetry, bilinearlity, and positivity). This function w is called a *weight function.*

- Symmetry: $\langle f,g\rangle = \int_{-L}^{L} fgw\,dx = \int_{-L}^{L} gfw\,dx = \langle g,f\rangle$
- Bilinearity: $\langle f,g+h\rangle = \int_{-L}^{L} f(g+h)w\,dx = \int_{-L}^{L}(fgw + fhw)dx = \int_{-L}^{L} fgw\,dx + \int_{-L}^{L} fhw\,dx$
  $= \langle f,g\rangle + \langle f,h\rangle$

  and $\langle f,cg\rangle = \int_{-L}^{L} f(cg)w\,dx = c\int_{-L}^{L} fgw\,dx = c\langle f,g\rangle$
- Positivity: $\langle f,f\rangle = \int_{-L}^{L} f^2w\,dx \geq 0$ since $(f^2w)(x) \geq 0$ for all x in $[-L,L]$

  and $\langle f,f\rangle = \int_{-L}^{L} f^2w\,dx = 0 \Leftrightarrow f(x) = 0$, since f^2w is piecewise continuous, and $w(x)>0$ for all x in $(-L,L)$

(2) The *Chebyshev polynomials* are a sequence of polynomials $\{T_0(x), T_1(x), T_2(x), \dots\}$ defined by the following trigonometric equation: $T_n(\cos(\theta)) = \cos(n\theta)$. For example, using the trig identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, we have $T_2(x) = 2x^2 - 1$.
What are $T_0(x)$ and $T_1(x)$ (these are easy)? What about $T_3(x)$ (this is harder)?
Remember your trig identities:

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$\cos(0\cdot\theta) = \cos(0) = 1$, so $T_0(x) = 1$.

$\cos(1\cdot\theta) = \cos(\theta)$, so $T_1(x) = x$

$\cos(3\theta) = \cos(2\theta+\theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)$

$= (2\cos^2(\theta)-1)\cos(\theta) - 2\sin^2(\theta)\cos(\theta)$

$= 2\cos^3(\theta) - \cos(\theta) - 2(1-\cos^2(\theta))\cos(\theta)$

$= 4\cos^3(\theta) - 3\cos(\theta)$, so $T_3(x) = 4x^3 - 3x$

(3) Show that the Chebyshev polynomials are orthogonal on the interval $[-1,1]$ with respect to the inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx$$

with weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$. What is $\langle T_0, T_0\rangle$? $\langle T_n, T_n\rangle$ for $n > 0$?

Hint: Use the trig substitution $x = \cos(\theta)$ and the defining property of the Chebyshev polynomials!

$$\langle T_n, T_m\rangle = \int_{-1}^{1} T_n(x)\,T_m(x)\frac{1}{\sqrt{1-x^2}}\,dx \qquad x=\cos\theta,\quad dx=-\sin\theta\, d\theta$$

$-1 \le x \le 1$ when $0 \le \theta \le \pi$

$$= \int_{\pi}^{0} T_n(\cos\theta)\,T_m(\cos\theta)\frac{-\sin\theta}{\sqrt{1-\cos^2\theta}}\,d\theta$$

$$= \int_0^{\pi} \underbrace{\cos(n\theta)\cos(m\theta)}_{\text{even Function}}\,d\theta$$

$= \pi$ if $n=m=0$ $\quad (\langle T_0, T_0\rangle = \pi)$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\cos(n\theta)\cos(m\theta)\,d\theta \longrightarrow = \frac{\pi}{2} \text{ if } n=m>0$$

$(\langle T_n, T_n\rangle = \frac{\pi}{2})$

$= 0$ if $n \neq m$

By our computation in class that $\cos(nx)$ and $\cos(mx)$ are orthogonal for the $\int_{-\pi}^{\pi} fg\,dx$ inner product on $C[-\pi,\pi]$.

(4) Find the first three terms in the Chebyshev series for $f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \end{cases}$.

$$f(x) \sim a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + \dots = 0T_0 + \frac{4}{\pi}T_1 + 0T_2 + \dots$$

$$a_0 = \frac{\langle f, T_0\rangle}{\langle T_0, T_0\rangle} = \frac{1}{\pi}\int_{-1}^{1} \underbrace{f(x)T_0(x)\frac{1}{\sqrt{1-x^2}}}_{\text{odd function}}dx = 0$$

$T_0 = 1$ is even and $w(x) = \frac{1}{\sqrt{1-x^2}}$ is even, $f(x)$ is odd

Similarly,

$$a_2 = \frac{\langle f, T_2\rangle}{\langle T_2, T_2\rangle} = \frac{2}{\pi}\int_{-1}^{1}\underbrace{f(x)(2x^2-1)\frac{1}{\sqrt{1-x^2}}}_{\text{odd}}dx = 0$$

$T_2 = 2x^2-1$ is even, $w(x) = \frac{1}{\sqrt{1-x^2}}$ is even, $f(x)$ is odd

$$a_1 = \frac{\langle f, T_1\rangle}{\langle T_1, T_1\rangle} = \frac{2}{\pi}\int_{-1}^{1}\underbrace{f(x)\,x\,\frac{1}{\sqrt{1-x^2}}}_{\text{even}}\,dx$$

$x=\cos\theta$, $dx = -\sin\theta\, d\theta$

$$= \frac{4}{\pi}\int_0^1 1\,x\frac{1}{\sqrt{1-x^2}}\,dx = \frac{4}{\pi}\int_{\pi/2}^{0}\cos\theta\frac{-\sin\theta}{\sqrt{1-\cos^2\theta}}\,d\theta = \frac{4}{\pi}\left(\sin\theta\Big|_0^{\pi/2}\right) = \frac{4}{\pi}$$

(5) *An extra open ended problem:* Making the change of variables $x = \cos(\theta)$, think about the relationship between the Chebyshev series for a function $g(x)$ defined on $[-1,1]$ and the Fourier cosine series for $g(\cos(\theta))$ defined on $[0,\pi]$...

$$f(x) \sim a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + \dots$$

$$f(\cos\theta) \sim a_0T_0(\cos\theta) + a_1T_1(\cos\theta) + a_2T_2(\cos(\theta)) + \dots$$

$$\sim a_0\cos(0\theta) + a_1\cos(1\theta) + a_2\cos(2\theta) + \dots$$

$$\sim a_0 + \sum_{n=1}^{\infty} a_n\cos(n\theta)$$