MATH 54 - WORKSHEET 13
THE HEAT EQUATION

In class, we analyzed the wave equation, found the form of a general solution given the boundary conditions, and found exact solutions for given initial conditions which could be expressed as (possibly infinite) sums of sine functions. On this worksheet, we'll do a similar analysis for the heat equation:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x)$$

The heat equation describes the diffusion of heat through a wire of length L. Here $u(x,t)$ is the amount of heat preset at position x along the wire at time t, with $0 \le x \le L$ and $t \ge 0$. For a derivation of this equation from physics, see Section 10.1 in the textbook.

Boundary conditons: For simplicity, we assume that the ends of the wire ($x = 0$ and $x = L$) are kept at a fixed heat level 0 at all times ($u(0,t) = u(L,t) = 0$). This condition is somewhat physically unrealistic, and in class next week we will consider what happens when we drop it.

Initial value condition: The function $f(x)$ describes the amount of heat present at position x along the wire at time $t = 0$ ($u(x,0) = f(x)$).

(1) First, we look for solutions of the form $u(x,t) = X(x)T(t)$. Assuming $u(x,t)$ can be written in this form, use the method of separation of variables to show that X and T satisfy the ordinary differential equations

$$X''(x) - kX(x) = 0$$
$$T'(t) - \beta k T(t) = 0.$$

If $u(x,t) = X(x)T(t)$, then $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$.
Then the heat equation is $XT' = \beta X''T$
Dividing both sides by X and βT, we get $T'/\beta T = X''/X$
Both sides must be equal to a constant K, so $T'/\beta T = K$ and $X''/X = K$
Rearranging, $T' = \beta K T \Rightarrow T' - \beta K T = 0$ and $X'' = KX \Rightarrow X'' - KX = 0$.

(2) Focusing on the X variable, we have the boundary value problem $X'' - kX = 0$, $X(0) = 0$, $X(L) = 0$. Note that this is the same boundary value problem we solved in class when we studied the wave equation! It has a solution if and only if $k = -\left(\frac{n\pi}{L}\right)^2$ for some integer n, and then the solution is

$$X(x) = C \sin\left(\frac{n\pi x}{L}\right).$$

Using this expression for k, find the general solution of the differential equation $T'(t) - \beta k T(t) = 0$.

We have $T' - \beta\left(-\left(\frac{n\pi}{L}\right)^2\right)T = 0$.

Aux. poly: $x + \beta\left(\frac{n\pi}{L}\right)^2$.
Root: $-\beta\left(\frac{n\pi}{L}\right)^2$.

General solution:
$Ce^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$

(3) Combine the general solutions for $X(x)$ and $T(t)$ in (2) into a general solution for $u(x,t)$. Taking the intial value condition $u(x,0) = f(x)$ into account now, what form must f have in order for $u(x,t)$ to be a solution?

$$u(x,t) = C\sin\left(\frac{n\pi}{L}x\right)e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

$$f(x) = u(x,0) = C\sin\left(\frac{n\pi}{L}x\right)e^{-\beta\left(\frac{n\pi}{L}\right)^2\cdot 0} = \underline{C\sin\left(\frac{n\pi}{L}x\right)}.$$

f must have this form for some integer n and some constant C.

(4) Check that if the functions $u_n(x,t)$ are all solutions of the form found in (3), then $\sum_{n=1}^{\infty} u_n(x,t)$ is a solution to the PDE and the boundary conditions (assuming it converges to a function with continuous second partial derivative).

If $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$, and each u_n has the form in (3), then each u_n is a solution to the PDE and boundary conditions.

$$\text{PDE: } \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} u_n(x,t)\right) = \sum_{n=1}^{\infty}\frac{\partial u_n}{\partial t} = \sum_{n=1}^{\infty}\beta\frac{\partial^2 u_n}{\partial x^2} = \beta\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty}u_n = \beta\frac{\partial^2 u}{\partial x^2}.$$

(since each u_n is a solution)

Boundary conditions: $u(0,t) = \sum_{n=1}^{\infty} u_n(0,t) = \sum_{n=1}^{\infty} 0 = 0$

$u(L,t) = \sum_{n=1}^{\infty} u_n(L,t) = \sum_{n=1}^{\infty} 0 = 0.$

(5) Solve the heat equation with $\beta = 3$, $L = \pi$, and $f(x) = \sin(x) + 3\sin(2x)$.

Take $C_1 = 1$ and $C_2 = 3$ and all other $C_n = 0$

$$u(x,t) = \sin(x)e^{-3t} + 3\sin(2x)e^{-12t}$$

(6) Find a formal solution to the heat equation with $\beta = 2$, $L = 1$, and $f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\pi x)$.

Take $C_n = \frac{1}{n^2}$ for all n.

$$u(x,t) = \sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\pi x)e^{-2n^2\pi^2 t}$$