MATH 54 - WORKSHEET 11
VANDERMONDE MATRICES AND LINEAR INDEPENDENCE

We found that if an order n homogeneous linear differential equation with constant coefficients has an auxiliary polynomial $p(x)$ with n distinct roots, $r_1, \ldots, r_n$, then the functions $e^{r_1 x}, \ldots, e^{r_n x}$ are solutions to the differential equation. We would like to show that these solutions are linearly independent.

There are many ways to do this (another approach is featured on your homework), but one way involves Vandermonde matrices, which are a neat linear algebra topic.

(1) Show that the Wronskian of the functions $e^{r_1 x}, \ldots, e^{r_n x}$ evaluated at 0, $W[e^{r_1 x}, \ldots, e^{r_n x}](0)$, is the $n \times n$ determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \ldots & r_n^{n-1} \end{vmatrix}.$$

The derivatives of $f(x)=e^{rx}$ are $f'(x)=re^{rx}$, $f''(x)=r^2e^{rx}, \ldots, f^{(n-1)}(x)=r^{n-1}e^{rx}$.

So,

$$W[e^{r_1x},\ldots,e^{r_nx}](x) = \begin{vmatrix} e^{r_1x} & e^{r_2x} & \cdots & e^{r_nx} \\ r_1e^{r_1x} & r_2e^{r_2x} & \cdots & r_ne^{r_nx} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1}e^{r_1x} & r_2^{n-1}e^{r_2x} & \cdots & r_n^{n-1}e^{r_nx} \end{vmatrix},$$

And plugging in 0,

$$W[e^{r_1x},\ldots,e^{r_nx}](0) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & & r_n^{n-1} \end{vmatrix}$$

(2) A matrix of the form in part (1) is called a Vandermonde matrix. We show below that its determinant is $\prod_{i<j}(r_j - r_i)$. This notation mean that for every pair of roots r_i and r_j, you take the difference $r_j - r_i$, and multiply up all these differences. For example, if $n = 3$, $\prod_{i<j}(r_j - r_i) = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2)$.

Assuming this fact for now, explain why this shows that the functions are linearly independent.

Since p has n distinct roots, all the numbers $r_1, \ldots, r_n$ are different. Then all the factors in $\prod_{i<j}(r_j - r_i)$ are nonzero, so

$W[e^{r_1x},\ldots,e^{r_nx}](0) \neq 0$. If the Wronskian is nonzero somewhere on the interval where the functions are solutions to an order n linear homogeneous diff. eq., then the functions are linearly independent.

(3) As a base case, check that the determinant of the 2×2 Vandermonde matrix $\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}$ is $(r_2 - r_1)$.

$$\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix} = 1 \cdot r_2 - 1 \cdot r_1 = (r_2 - r_1)$$

(4) Now consider the $(n+1) \times (n+1)$ determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ r_1 & r_2 & \dots & r_n & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} & x^{n-1} \\ r_1^n & r_2^n & \dots & r_n^n & x^n \end{vmatrix}.$$

I have replaced r_{n+1} with x so that we think of it as a variable. Explain why the determinant is a polynomial in x of degree at most n (Hint: Think about the expansion down the last column). Can you find n roots to this polynomial? (Hint: What happens when two columns of a matrix are equal?).

Expanding down the last column, we get a sum of powers of x times determinants of $n \times n$ matrices. These matrices have constant entries, and the highest power of x appearing is x^n, so the result is a polynomial of degree $\leq n$ in x.

Roots: $r_1, \dots, r_n$. If x takes on one of these values, two columns of the matrix will be equal, so the determinant will be 0.

(5) Conclude that the determinant in part (4) can be factored as $C(x - r_n)(x - r_{n-1}) \dots (x - r_1)$ for some constant C. Note that C is the coefficient of x^n. Can you express C as the determinant of a smaller Vandermonde matrix?

Any polynomial of degree $\leq n$ with n roots $r_1, \dots, r_n$ factors this way. Expanding, C is the coefficient of x^n.

Expanding the determinant along the last column, we see that the coefficient of x^n is

$$C = \begin{vmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{vmatrix}.$$

This is the determinant of a smaller $(n \times n)$ Vandermonde matrix, so its determinant is $\prod_{1 \leq i < j \leq n} (r_j - r_i)$

(6) Finally, replace x by r_{n+1}, and show that the determinant of the matrix in part (4) (with x replaced by r_{n+1}) is equal to $\prod_{i<j} (r_j - r_i)$, as we wanted.

We found that the determinant is

$$C(x-r_1)\dots(x-r_n) = \left(\prod_{1 \leq i < j \leq n} (r_j - r_i)\right)(x-r_1)\dots(x-r_n).$$

Replacing x with r_{n+1}, we have

$$= \left(\prod_{1 \leq i < j \leq n} (r_j - r_i)\right)(r_{n+1}-r_1)\dots(r_{n+1}-r_n)$$

$$= \prod_{1 \leq i < j \leq n+1} (r_j - r_i).$$