MATH 54 - WORKSHEET 10
LEAST-SQUARES SOLUTIONS

(1) Sometimes linear equations are *inconsistent*, i.e. they have no solutions. Row reduce to check that the matrix equation $A\mathbf{x} = \mathbf{b}$ below is inconsistent.

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

inconsistent

(2) In these situations, we'd like to find a vector $\mathbf{x}$ such that $A\mathbf{x}$ is *as close as possible* to $\mathbf{b}$. That is, we'd like to find $\mathbf{x}$ minimizing the distance $||A\mathbf{x} - \mathbf{b}||$. Explain why to minimize this length, we should try to solve the equation $A\mathbf{x} = \text{proj}_{\text{Col}(A)}\mathbf{b}$. A solution to this equation is called a **least-squares solution** or a **best approximation** to $A\mathbf{x} = \mathbf{b}$.

The Column space of A is the set of all vectors of the form $A\bar{x}$.
$\text{proj}_{\text{Col}(A)}\bar{b}$ is the vector in Col(A) which is the closest to $\bar{b}$.
So to minimize $||A\bar{x} - \bar{b}||$, we must find $\bar{x}$ so that $A\bar{x} = \text{proj}_{\text{Col}(A)}\bar{b}$.

(3) In class, we used Gram-Schmidt to find an orthogonal basis for Col(A), with A as in (1):

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} 1/5 \\ 1/5 \\ -3/5 \\ 2/5 \end{pmatrix}.$$

Use this basis C to compute $\text{proj}_{\text{Col}(A)}\mathbf{b}$ and solve the equation in (2).

$$\text{proj}_{\text{Col}(A)}\bar{b} = \frac{\bar{b}\cdot\bar{c}_1}{\bar{c}_1\cdot\bar{c}_1}\bar{c}_1 + \frac{\bar{b}\cdot\bar{c}_2}{\bar{c}_2\cdot\bar{c}_2}\bar{c}_2 + \frac{\bar{b}\cdot\bar{c}_3}{\bar{c}_3\cdot\bar{c}_3}\bar{c}_3$$

$$= \frac{1}{2}\bar{c}_1 + \frac{7/2}{5/2}\bar{c}_2 + \frac{2/5}{3/5}\bar{c}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 7/10 \\ 7/10 \\ 7/5 \\ 7/5 \end{pmatrix} + \begin{pmatrix} 2/15 \\ 2/15 \\ -2/5 \\ 4/15 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1/3 \\ 1 \\ 5/3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4/3 \\ -1 & 1 & 0 & 1/3 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 5/3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4/3 \\ 0 & 1 & 1 & 5/3 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 5/3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4/3 \\ 0 & 1 & 1 & 5/3 \\ 0 & 0 & -1 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \bar{x} = \begin{pmatrix} 2/3 \\ 1 \\ 2/3 \end{pmatrix}$$

Unfortunately, the strategy used above to compute least-squares solutions (use Gram-Schmidt to find an orthogonal basis for Col(A), use this basis to project $\mathbf{b}$ orthogonally onto Col(A), then solve the equation $A\mathbf{x} = \text{proj}_{\text{Col}(A)}\mathbf{b}$) is a lot of work! The next two problems outline a cleverer strategy.

(4) Show that $\hat{\mathbf{x}}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$ (i.e. $A\hat{\mathbf{x}} = \text{proj}_{\text{Col}(A)}\mathbf{b}$) if and only if $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$, and conclude that the least-squares solutions to $A\mathbf{x} = \mathbf{b}$ are the solutions to $A^TA\mathbf{x} = A^T\mathbf{b}$. [*Hint:* Recall that the orthogonal complement of Col(A) is Nul(A^T).]

$\hat{x}$ is a least-squares solution to $A\bar{x} = \bar{b}$ $\iff$ $A\hat{x} = \text{proj}_{\text{Col}(A)}\bar{b}$

Here's a picture for this one

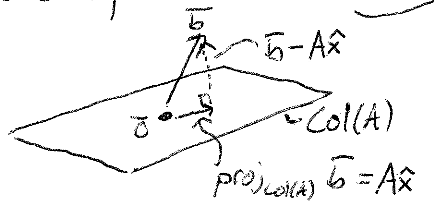
$\iff$ $\bar{b} - A\hat{x}$ is orthogonal to Col(A)

$\iff$ $\bar{b} - A\hat{x}$ is in Nul(A^T)

$\iff$ $A^T(\bar{b} - A\hat{x}) = 0$

$\iff$ $A^T\bar{b} - A^TA\hat{x} = \bar{0}$

$\iff$ $A^TA\hat{x} = A^T\bar{b}$.



(5) Use the characterization of least-squares solution in (4) to find the least-squares solution to the equation $A\mathbf{x} = \mathbf{b}$ in (1). [Your answer should be the same as in (3)]

We solve $A^TA\bar{x} = A^T\bar{b}$:

$$A^TA = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A^T\bar{b} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

Row reduce

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ -1 & 3 & 1 & 3 \\ 1 & 1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & 5/2 & 3/2 & 7/2 \\ 0 & 3/2 & 3/2 & 5/2 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & 5 & 3 & 7 \\ 0 & 3 & 3 & 5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 6/5 & 4/5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & 2/3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 1/6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \end{array}\right) \qquad \bar{x} = \begin{pmatrix} 2/3 \\ 1 \\ 2/3 \end{pmatrix} \checkmark$$