# MATH 54 - WORKSHEET 1
# MONDAY 6/22

**Row Operations:**

(1) (Replacement) Add a multiple of one row to another row.
(2) (Interchange) Swap two rows.
(3) (Scaling) Multiply an entire row by a nonzero constant.

A matrix is in **echelon form** if:

- All rows with only 0s are on the bottom.
- The leading entry (first nonzero entry) of each row is to the right of the leading entry of all rows above it.
- Only 0s appear *below* the leading entry of each row.

**Example:**

$$\begin{pmatrix} 0 & 1 & 1 & 7 & 1 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A matrix is in **reduced echelon form** if, additionally:

- All leading entries are 1.
- Only 0s appear *above* the leading entry of each row.

**Example:**

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Problems:**

(1) Consider the following system of linear equations:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 3 \\ 4x_2 + x_3 &= 6 \end{aligned}$$

(a) Convert the system to an augemented matrix.

**Answer:**

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 0 & 4 & 1 & 6 \end{array}\right)$$

(b) Apply row operations to put the matrix in echelon form.

**Answer:**

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 4 & 1 & 6 \end{array}\right) \qquad \text{row}_2 \leftarrow \text{row}_2 - \text{row}_1$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 14 \end{array}\right) \qquad \text{row}_3 \leftarrow \text{row}_3 + 4 \cdot \text{row}_2$$

(c) Further apply row operations to put the matrix in reduced echelon form.

**Answer:**

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 5 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 14 \end{array}\right) \qquad \text{row}_1 \leftarrow \text{row}_1 + 2 \cdot \text{row}_2$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & -9 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 14 \end{array}\right) \qquad \text{row}_1 \leftarrow \text{row}_1 - \text{row}_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{9}{2} \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 14 \end{array}\right) \qquad \text{row}_1 \leftarrow \frac{1}{2} \cdot \text{row}_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{9}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 14 \end{array}\right) \qquad \text{row}_2 \leftarrow -\text{row}_2$$

(d) Find the solutions to the system of linear equations.

**Answer:**

$$\begin{aligned} x_1 &= -\frac{9}{2} \\ x_2 &= -2 \\ x_3 &= 14 \end{aligned}$$

(2) The solutions to a system of linear equations are *invariant under row operations*. That is, if we start with a system of linear equations, convert it to an augmented matrix, apply some row operations, and then convert back to a system of linear equations, the new system has the same solutions as the old system. Explain why.

**Answer:** I will describe why the solutions to a system are invariant under each of the row operations.

(a) Interchange: Swapping two rows of the matrix corresponds to swapping two equations in the system. We have the same collection of equations, just listed in a different order, so the solutions did not change.

(b) Scaling: Scaling a row by a nonzero constant corresponds to multiplying both sides of an equation by a nonzero constant. Since the solutions to the equation $a_1x_1 + \cdots + a_nx_n = b$ are the same as the solutions to the equation $c(a_1x_1 + \cdots + a_nx_n) = cb$, and the other equations in the system did not change, the solutions to the system did not change.

(c) Replacement: This operation corresponds to taking two equations $a_1x_1 + \cdots + a_nx_n = b$ and $a'_1x_1 + \cdots + a'_nx_n = b'$ and replacing the latter by

$$(ca_1 + a'_1)x_1 + \cdots + (ca_n + a'_n)x_n = cb + b'.$$

Let's say you give me a solution to the original system $x_1 = d_1, \ldots, x_n = d_n$. Then it is also a solution to the new system. The only equation we have to check is the new one, which we can equivalently write as

$$c(a_1x_1 + \cdots + a_nx_n) + a'_1x_1 + \cdots + a'_nx'_n = cb + b'.$$

And since $x_1 = d_1, \ldots, x_n = d_n$ is a solution to the old system, we have

$$a_1d_1 + \cdots + a_nd_n = b$$

and

$$a'_1d_1 + \cdots + a'_nd_n = b',$$

so

$$c(a_1d_1 + \cdots + a_nd_n) + a'_1d_1 + \cdots + a'_nd'_n = cb + b'.$$

Ok, so every solution to the old system is a solution to the new system. How do we know that we didn't accidentally introduce any new solutions? That is, how do we know that every solution to the new system is a solution to the old system? Well, we can get from the new system back to the old system by reversing the replacement operation we just did. That is, if we did the replacement operation

$$\text{row}_i \leftarrow \text{row}_i + c \cdot \text{row}_j,$$

we can get back to the original matrix by

$$\text{row}_i \leftarrow \text{row}_i + (-c) \cdot \text{row}_j.$$

Then exactly the same argument as above shows that every solution of the new system is a solution of the old system.

(3) We could also define column operations on a matrix (e.g. add a multiple of one column to another column). Are the solutions to a system of linear equations invariant under column operations? Explain why or why not.

**Answer:** No, they aren't. For example, consider the following augmented matrix, corresponding to the single equation $x_1 + 2x_2 = 0$:

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \end{array}\right)$$

Interchanging columns corresponds to swapping the names of the variables. If we swap the first and the second column, we get

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \end{array}\right),$$

which corresponds to the equation $2x_1 + x_2 = 0$. This has a different set of solutions from the original equation. For example, $x_1 = 2$, $x_2 = -1$ is a solution to the original equation, but not the new one.

Also, scaling a column produces a totally different equation. If we scale the first column by 5, we get

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \end{array}\right),$$

which corresponds to the equation $5x_1+2x_2 = 0$, which has a different set of solutions.

(4) The system in problem (1) is **consistent** (i.e. it has a solution) and the solution is **unique**.

(a) How could you tell this just by looking at the echelon form?

**Answer:** In the echelon form, the column after the vertical line doesn't contain a pivot, so the system is consistent (the reduced echelon form will not include the equation $0 = 1$). And every column before the vertical line contains a pivot, so the solution is unique (there are no free variables).

(b) For each of the following augmented matrices in echelon form, determine *without doing any calculations* whether the corresponding system of linear equations is consistent, and if it is, whether the solution is unique.

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 7 & 1 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

**Answer:** Inconsistent.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

**Answer:** Consistent with a unique solution.

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

**Answer:** Consistent with infinitely many solutions. (The first variable $x_1$ is free. If you write down the equations, you'll see that $x_1$ isn't even mentioned, which means it can take on any value.)

$$\left(\begin{array}{c|c} 1 & 0 \end{array}\right)$$

**Answer:** Consistent with a unique solution.

$$\left(\begin{array}{c|c} 0 & 0 \end{array}\right)$$

**Answer:** Consistent with infinitely many solutions.

$$\left(\begin{array}{c|c} 0 & 1 \end{array}\right)$$

**Answer:** Inconsistent.