Name: Solutions

MIDTERM 2 - JULY 24, 2015

MATH 54 SECTION 8 - ALEX KRUCKMAN

Please put away everything except scratch paper and pencils/pens.
This exam begins at 8:10am and ends at 10am. You have 110 minutes.
Write your answers, *including complete justifications*, in the spaces provided.
If you finish early or have a question, please make your way quietly to the front of the room, taking care not to disturb the other test takers!

Problem	Out of	Score
1	10	
2	12	
3	12	
4	5	
5	9	
6	12	
Total:	60	

Scores on this exam

HW	Out of	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total:	105	

Homework scores

Quiz	Out of	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Quiz scores

(1) For this problem, consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

(a) (4 points) Find all eigenvalues of A.

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -2-\lambda \\ 1 & -1 \end{vmatrix}$$

$$= -\lambda((-2-\lambda)(-\lambda)+1) + (-\lambda-1) + (-1+2+\lambda)$$

$$= -\lambda(\lambda^2+2\lambda+1)$$

$$= -\lambda(\lambda+1)^2$$

Eigenvalues: $0, -1$

(b) (6 points) Find a basis B such that $[A]_B$ is diagonal.

0-eigenspace:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = x_3$
$x_2 = x_3$
x_3 free

$$\bar{x} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

-1-eigenspace:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = x_2 - x_3$
x_2, x_3 free

$$\bar{x} = \begin{pmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Basis: $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$[A]_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(2) For this problem, consider the subspace of $\mathbb{R}^3$

$$W = \text{Span}\left\{ \underset{\overline{b_1}}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}, \underset{\overline{b_2}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \right\}$$

:

(a) (4 points) Use the Gram-Schmidt process to find an orthogonal basis for W.

- $\overline{c_1} = \overline{b_1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$
- $\overline{c_2} = \overline{b_2} - \dfrac{\overline{b_2} \cdot \overline{c_1}}{\overline{c_1} \cdot \overline{c_1}} \overline{c_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \dfrac{3}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -1/5 \\ 1 \end{pmatrix}$

Basis: $\left\{ \underset{\overline{c_1}}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}, \underset{\overline{c_2}}{\begin{pmatrix} 2/5 \\ -1/5 \\ 1 \end{pmatrix}} \right\}$

(b) (4 points) Find the orthogonal projection of $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 1/5 \end{pmatrix}$ onto W.

$$\text{proj}_W \overline{v} = \frac{\overline{v} \cdot \overline{c_1}}{\overline{c_1} \cdot \overline{c_1}} \overline{c_1} + \frac{\overline{v} \cdot \overline{c_2}}{\overline{c_2} \cdot \overline{c_2}} \overline{c_2}$$

$$= \frac{5}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{6/5}{6/5}\begin{pmatrix} 2/5 \\ -1/5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7/5 \\ 9/5 \\ 1 \end{pmatrix}$$

$\overline{v} \cdot \overline{c_2} = 3(2/5) + 1(-1/5) + 1/5(1) = 6/5$

$\overline{c_2} \cdot \overline{c_2} = (2/5)^2 + (-1/5)^2 + 1^2$
$= 4/25 + 1/25 + 25/25$
$= 30/25 = 6/5$

(c) (4 points) Find a basis for the orthogonal complement $W^{\perp}$.

$W^{\perp}$ is the nullspace of a matrix whose rowspace is W

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$x_1 = -2x_3$
$x_2 = x_3$
x_3 free

$\overline{x} = x_3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ so $\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$ is a basis for $W^{\perp}$.

Alternatively, since $\dim(W) + \dim(W^{\perp}) = 3 \Rightarrow \dim(W^{\perp}) = 1$, we only need one nonzero vector in $W^{\perp}$. For example,

$\overline{v} - \text{proj}_W \overline{v} = \begin{pmatrix} 3 \\ 1 \\ 1/5 \end{pmatrix} - \begin{pmatrix} 7/5 \\ 9/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 8/5 \\ -4/5 \\ -4/5 \end{pmatrix}$ works.

(3) Recall that $C[0,1]$, the space of all continuous functions $[0,1] \to \mathbb{R}$, is an inner product space, with inner product $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$. Use this inner product in the definitions of length, angle, and orthogonality in (a) - (c).

(a) (3 points) Find the length $||f(x)||$ of the function $f(x) = x$.

$$||f|| = \sqrt{<f,f>} = \sqrt{\int_0^1 x^2\,dx} = \sqrt{\tfrac{1}{3}x^3\Big|_0^1} = \sqrt{1/3}$$

(b) (3 points) Find the angle between the functions $f(x) = x$ and $g(x) = 1$.

$$\cos\theta = \frac{<f,g>}{||f||\,||g||} = \frac{\int_0^1 x\,dx}{\sqrt{1/3}\,\sqrt{\int_0^1 1\,dx}} = \frac{\frac{1}{2}x^2\Big|_0^1}{\sqrt{1/3}\,\sqrt{x\Big|_0^1}} = \frac{1/2}{1/\sqrt{3}\cdot 1} = \frac{\sqrt{3}}{2}$$

from (1)

So $\theta = \pi/6$ or $30°$

(c) (3 points) Find a nonzero function which is orthogonal to the function $g(x) = 1$.

We want to find $h(x)$ with the property that $<h,g> = \int_0^1 h(x)\,dx = 0$.

For example, $x - \frac{1}{2}$ works. $\int_0^1 x - \frac{1}{2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}x\Big|_0^1 = \frac{1}{2} - \frac{1}{2} = 0$.

(d) (3 points) Show that for all functions $f(x)$ in $C[0,1]$,

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \int_0^1 f(x)^2\,dx.$$

Hint: Use the Cauchy-Schwarz inequality $|\langle \mathbf{u}, \mathbf{v}\rangle| \le ||\mathbf{u}||\,||\mathbf{v}||$.

Applying Cauchy-Schwarz with $\vec{u} = f(x)$ and $\vec{v} = g(x) = 1$,

$|<f,g>| \le ||f||\,||g||$

$\left|\int_0^1 f(x)g(x)\,dx\right| \le \sqrt{\int_0^1 f(x)^2\,dx}\,\sqrt{\int_0^1 g(x)^2\,dx}$, but $g(x) = 1$, so

$\left|\int_0^1 f(x)\,dx\right| \le \sqrt{\int_0^1 f(x)^2\,dx}$ squaring both sides,

$\left(\int_0^1 f(x)\,dx\right)^2 \le \int_0^1 f(x)^2\,dx$

(4) (5 points) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in an inner product space V. Show that if $\langle \mathbf{u}, \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all vectors $\mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.

For all $\bar{w}$, $\langle \bar{u}, \bar{w}\rangle = \langle \bar{v}, \bar{w}\rangle \Rightarrow \langle \bar{u}, \bar{w}\rangle - \langle \bar{v}, \bar{w}\rangle = 0$

$\Rightarrow \langle \bar{u} - \bar{v}, \bar{w}\rangle = 0$

But taking $\bar{w} = \bar{u} - \bar{v}$, we have

$\langle \bar{u}-\bar{v}, \bar{u}-\bar{v}\rangle = 0$

$\Rightarrow \bar{u} - \bar{v} = 0$ by positivity

$\Rightarrow \bar{u} = \bar{v}$.

For partial credit: If $\langle \bar{x}, \bar{y}\rangle$ is dot product on $\mathbb{R}^n$, then for all $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\langle \bar{x}, \bar{e}_i\rangle = x_i$.

So if $\underbrace{\langle \bar{u}, \bar{e}_i\rangle}_{u_i} = \underbrace{\langle \bar{v}, \bar{e}_i\rangle}_{v_i}$ for all i, then $\bar{u}$ and $\bar{v}$ have the same entries, and $\bar{u} = \bar{v}$.

(5) For each of the following, give a *short* explanation:

(a) (3 points) The matrices A and C below are not similar. Why not?

$$A = \begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

They have different eigenvalues and determinants.

(b) (3 points) No 3×3 matrix with real entries has $1 + i$ and $2 - 3i$ as complex eigenvalues. Why not?

The e.values of a real matrix come in complex conjugate pairs. So $1-i$ and $2+3i$ are also eigenvalues. But a 3×3 matrix can't have 4 eigenvalues.

(c) (3 points) No 3×3 *symmetric* matrix B with real entries satisfies the equations

$$B\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ and } B\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Why not?

Any two eigenvectors of a symmetric matrix with different eigenvalues are orthogonal.

But $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3 \neq 0$.

(6) For this problem, you do *not* need to explain your answers.

Consider the matrix

$$A = \begin{pmatrix} \sqrt{1/3} & \sqrt{2/3} \\ c & \sqrt{1/3} \end{pmatrix}.$$

Find values of c which make A

(a) (2 points) symmetric

$\sqrt{2/3}$

(b) (2 points) orthogonal

$-\sqrt{2/3}$

Here are some explanations for your benefit:

(c) (2 points) singular

$\frac{1}{\sqrt{2}\sqrt{3}}$

$|A| = \sqrt{1/3}\sqrt{1/3} - c\sqrt{2/3}$

Solving $|A|=0$ for c, $c\sqrt{2/3} = 1/3$

$\Rightarrow c = \frac{1}{3}\cdot\frac{\sqrt{3}}{\sqrt{2}} = \frac{1}{\sqrt{2}\sqrt{3}}$.

(d) (2 points) diagonalizable over $\mathbb{R}$

$\sqrt{2/3}$ — Real symmetric matrices are always diagonalizable over $\mathbb{R}$

or $\frac{1}{\sqrt{2}\sqrt{3}}$ — If A is singular, it has 0 as an e.value. But it also has a nonzero e.value $\Rightarrow$ 2 distinct e.values $\Rightarrow$ diagonalizable.

Alternatively, note that $|A-\lambda I| = \lambda^2 - 2/\sqrt{3}\,\lambda + \frac{1}{3} - \sqrt{\frac{2}{3}}c$

so the e.values are $\frac{2/\sqrt{3} \pm \sqrt{(-2/\sqrt{3})^2 - 4(\frac{1}{3} - \sqrt{\frac{2}{3}}c)}}{2} = \frac{1}{\sqrt{3}} \pm \sqrt{\sqrt{\frac{2}{3}}c}$

If $c>0$, there are 2 distinct real e.values $\Rightarrow$ diagonalizable $/\mathbb{R}$

If $c<0$, there are 2 distinct complex e.values $\Rightarrow$ diag. $/\mathbb{C}$ but not $/\mathbb{R}$.

(e) (2 points) diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$

$-\sqrt{2/3}$ — The matrix then has the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, So it has 2 distinct complex e.values, $\sqrt{1/3} \pm i\sqrt{2/3}$.

(f) (2 points) not diagonalizable even over $\mathbb{C}$

0 — The matrix $\begin{pmatrix} \sqrt{1/3} & \sqrt{2/3} \\ 0 & \sqrt{1/3} \end{pmatrix}$ has only one e.value ($\sqrt{1/3}$)

but the $\sqrt{1/3}$-e.space is only one-dimensional:

$A - \sqrt{1/3}I = \begin{pmatrix} 0 & \sqrt{2/3} \\ 0 & 0 \end{pmatrix}$, rank 1.