Name: ____________

MATH 54 MIDTERM 2
SUMMER 2011 - SECTION 5 - ALEX KRUCKMAN

Please put away everything except scratch paper and pencils/pens.
You have 110 minutes to complete this exam, which ends at 2pm sharp.
Write your answers, *including complete justifications*, in the spaces provided below.
If you finish early or have a question, please make your way to the front of the room, taking care not to disturb the other test takers!

(1) (12 points) Consider the matrix:

$$A = \begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \\ -1 & -2 & -4 & 1 & -3 \\ 1 & 2 & 5 & 5 & 4 \\ -2 & -4 & -7 & 1 & -5 \end{pmatrix}.$$

(a) Find a basis for Row(A).
(b) Find a basis for Col(A).
(c) Find a basis for Nul(A).

Row reduce: $A \longrightarrow \begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 7 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 5 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

a) Row A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ (pivot rows)

b) Col A: $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ -4 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \\ 5 \\ 1 \end{pmatrix} \right\}$ (pivot colums)

c) If $\bar{x}$ is in Nul(A), $\bar{x} = \begin{pmatrix} -2x_2 + x_5 \\ x_2 \\ -x_5 \\ 0 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. Nul A: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(2) (12 points) Suppose that the characteristic polynomial of a matrix A is $\lambda^3 - 2\lambda^2 + 2\lambda$. For each question below: answer the question or write "not enough information given" and *explain*.

(a) If A is $m \times n$, what are m and n?
(b) Is A invertible?
(c) Is there a vector $\overline{v}$ such that $A\overline{v} = \overline{v}$?
(d) Is A diagonalizable over $\mathbb{R}$?
(e) Is A diagonalizable over $\mathbb{C}$?
(f) What is dim(Row(A))?

The characteristic polynomial factors as $\lambda(\lambda^2 - 2\lambda + 2)$. It has roots 0 and $\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = 1 \pm \frac{\sqrt{-4}}{2} = 1 \pm i$.

a) $m = n = 3$. The characteristic polynomial only makes sense for $n \times n$ matrices, in which case it has degree n.

b) No. A has 0 as an eigenvalue, so it is singular.

c) No. If such a $\overline{v}$ existed, it would be an eigenvector of A with eigenvalue 1. But 1 is not an eigenvalue of A.

d) No. The characteristic polynomial of A has complex roots. Any diagonal matrix similar to A cannot have all real entries.

e) Yes. A has 3 distinct eigenvalues in $\mathbb{C}$, so it has 3 linearly independent eigenvectors and is diagonalizable.

f) dim(Row(A)) = 2. Since 0 appears as a single root of $|A - \lambda I|$, the dimension of Nul(A), the 0 eigenspace, is at most 1.
So dim(Row(A)) = rank(A) = 3 − dim(Nul(A)) = 2.

And in fact it's exactly 1, since 0 is an eigenvalue!

(3) (8 points) Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation given by $T(p(x)) = x\left(\frac{d}{dx}p(x)\right)$. You do not need to show that T is a linear transformation.

(a) Find the coordinate matrix for T with respect to the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for $\mathbb{P}_3$. This is denoted $[T]_\mathcal{B}$ or $[T]_{\mathcal{B}\mathcal{B}}$.

(b) What are the eigenvalues of $[T]_\mathcal{B}$? For each eigenvalue, give an eigenvector in $\mathbb{R}^4$ and the corresponding polynomial in $\mathbb{P}_3$.

a) $[T]_\mathcal{B} = \left([T(b_1)]_\mathcal{B}\ [T(b_2)]_\mathcal{B}\ [T(b_3)]_\mathcal{B}\ [T(b_4)]_\mathcal{B}\right)$

$T(1) = x \cdot 0 = 0$ $\quad [T(1)]_\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$T(x) = x \cdot 1 = x$ $\quad [T(x)]_\mathcal{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$T(x^2) = x(2x) = 2x^2$ $\quad [T(x^2)]_\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$

$T(x^3) = x(3x^2) = 3x^3$ $\quad [T(x^3)]_\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$

So $[T]_\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$,

b) Eigenvalues: 0, 1, 2, 3.

$0: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow 1$

$1: \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow x$

$2: \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightsquigarrow x^2$

$3: \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow x^3$

eigenvectors corresponding polynomials

Note that $[T]_\mathcal{B}$ is diagonal, so the standard basis vectors are eigenvectors.

(4) (8 points) Let V be a vector space with a basis $\{\overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4\}$, and let W be a subspace of V which is not the subspace $\{\overline{0}\}$.

(a) Explain why W has a basis.

(b) Suppose $\overline{v}_1$, $\overline{v}_2$, and $\overline{v}_3$ are in W, but $\overline{v}_4$ is not in W. Explain why $\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$ is a basis for W.

a) W is a subspace of V and V is finite dimensional, so W is finite-dimensional and $\dim(W) \leq \dim(V)$. In particular, since W has a finite dimension, it has a basis.

Alternative answer: Since $W \neq \{\overline{0}\}$, there is some nonzero $\overline{w}$ in W. $S = \{\overline{w}\}$ is a linearly independent set (since $\overline{w} \neq \overline{0}$), so by the expansion theorem, some expansion of S is a basis for W.

b) $\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$ is linearly independent, since it is part of a basis for V. Since W contains 3 linearly independent vectors, $\dim(W) \geq 3$. But also $\dim(W) < 4$, since $\overline{v}_4$ is not in W, so $W \neq V$. Hence $\dim(W) = 3$. We conclude: Any linearly independent set of 3 vectors in W is a basis, so $\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$ is a basis for W.

(5) (10 points) Consider the matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \end{pmatrix}.$$

(a) List the eigenvalues of A and their algebraic multiplicities.

(b) If A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$ (you are not required to show that P is invertible or to find P^{-1}). If A is not diagonalizable, explain why not.

(c) Compute A^{42}.

(d) Find [redacted] B such that $AB = A$. **Hint:** There is a way to do this without any further calculations.

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & -3-\lambda & 2 \\ 0 & -3 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -3-\lambda & 2 \\ -3 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)((-3-\lambda)(2-\lambda)+6) = (1-\lambda)(\lambda^2+\lambda)$$

$$= \lambda(1-\lambda)(\lambda+1).$$

a) Eigenvalues: $0, 1, -1$. All multiplicity 1.

b) 0-eigenvalue: $A \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix}$

$\bar{x} = \begin{pmatrix} 1/3\, x_3 \\ 2/3\, x_3 \\ x_3 \end{pmatrix}$. Taking $x_3 = 3$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. x_3 free

1-eigenvalue: $A - I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -4 & 2 \\ 0 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\bar{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$ Taking $x_1 = 1$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ x_1 free

(-1)-eigenvalue: $A + I = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\bar{x} = \begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix}$. Taking $x_3 = 1$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. x_3 free

$P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

We had $P=\begin{pmatrix}1&1&0\\2&0&1\\3&0&1\end{pmatrix}$. Finding P^{-1}: $\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\2&0&1&0&1&0\\3&0&1&0&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&-2&1&-2&1&0\\0&-3&1&-3&0&1\end{array}\right)$

$$\to\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&-\frac{1}{2}&1&-\frac{1}{2}&0\\0&-3&1&-3&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&-\frac{1}{2}&1&-\frac{1}{2}&0\\0&0&-\frac{1}{2}&0&-\frac{3}{2}&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&0&1&1&-1\\0&0&1&0&3&-2\end{array}\right)$$

$$\to\left(\begin{array}{ccc|ccc}1&0&0&0&-1&1\\0&1&0&1&1&-1\\0&0&1&0&3&-2\end{array}\right)$$

c) $A^{42}=PD^{42}P^{-1}=P\begin{pmatrix}0^{42}&0&0\\0&1^{42}&0\\0&0&(-1)^{42}\end{pmatrix}P^{-1}=\begin{pmatrix}1&1&0\\2&0&1\\3&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&1\\1&1&-1\\0&3&-2\end{pmatrix}$

$$=\begin{pmatrix}0&1&0\\0&0&1\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&1\\1&1&-1\\0&3&-2\end{pmatrix}=\begin{pmatrix}1&1&-1\\0&3&-2\\0&3&-2\end{pmatrix}$$

d) If k is odd, $A^k=PD^kP^{-1}=PDP^{-1}=A$,

since $0^k=0$, $(-1)^k=-1$, and $1^k=1$.

So any even power j of A satisfies $AA^j=A^{j+1}=A$.

In particular, we may take $B=A^{42}=\begin{pmatrix}1&1&-1\\0&3&-2\\0&3&-2\end{pmatrix}$