Vojta Midterm 2 (Spring '01) Solutions

1. Row reduce A.

$$A \to \begin{pmatrix} 1 & 6 & 2 & -3 & -4 \\ 0 & 0 & 4 & 11 & 19 \\ 0 & 0 & 4 & 11 & 19 \\ 0 & 0 & 2 & 7 & 3 \\ 0 & 0 & 0 & -3 & 13 \end{pmatrix} \to \begin{pmatrix} 1 & 6 & 2 & -3 & -4 \\ 0 & 0 & 1 & 11/4 & 19/4 \\ 0 & 0 & 0 & 3/2 & -13/2 \\ 0 & 0 & 0 & -3 & 13 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 6 & 2 & -3 & -4 \\ 0 & 0 & 1 & 11/4 & 19/4 \\ 0 & 0 & 0 & 1 & -13/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a). A basis for Row(A) is

$$\left\{ \begin{pmatrix} 1 \\ 6 \\ 2 \\ -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 11/4 \\ 19/4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -13/3 \end{pmatrix} \right\}$$

b) A basis for Col(A) is

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}, \begin{pmatrix} 10 \\ 2 \\ 8 \\ 0 \\ 12 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \\ 10 \\ -21 \end{pmatrix} \right\}$$

<u>Note</u>: Different row-reduction steps may produce different bases for Row(A). Of course any vector space has many possible bases, so this question has many possible answers.

2. We wish to solve

$$c_1 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} + c_3 \begin{pmatrix} -3 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix} \quad \text{for } c_1, c_2, c_3.$$

To do this, row reduce the augmented matrix

$$\left(\begin{array}{ccc|c} -1 & 2 & -3 & 1 \\ 3 & -4 & 8 & 2 \\ 1 & -3 & 2 & -8 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 2 & -1 & 5 \\ 0 & -1 & -1 & -7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -3 & -9 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 0 & -10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

So $c_1 = -2$, $c_2 = 4$, $c_3 = 3$, $\begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix} = -2\bar{b}_1 + 4\bar{b}_2 + 3\bar{b}_3$, and $\left[\begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$.

3. We haven't covered this.

4. Or this.

5. I'll do this by row-reduction. Remember, replacement operations are "free".

$$|A| = \begin{vmatrix} 1 & 2 & 7 & -5 \\ 0 & 2 & -3 & 11 \\ 0 & 2 & 1 & 8 \\ 0 & 4 & -1 & 23 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 7 & -5 \\ 0 & 2 & -3 & 11 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 5 & 1 \end{vmatrix} = 1\cdot 2\cdot \begin{vmatrix} 4 & -3 \\ 5 & 1 \end{vmatrix}$$

$$= 2\cdot(4-(-15)) = 2\cdot 19 = 38.$$

6. a)

$$A-3I = \begin{pmatrix} 0 & 0 & 0 \\ -6 & 0 & -2 \\ 6 & 0 & 2 \end{pmatrix},$$ which row reduces to $$\begin{pmatrix} -6 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\bar{x}$ in Nul$(A-3I)$ looks like $\bar{x} = \begin{pmatrix} -\frac{1}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix}$. So the 3-eigenspace has as a basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

b)

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -6 & 3-\lambda & -2 \\ 6 & 0 & 5-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ 0 & 5-\lambda \end{vmatrix} = (3-\lambda)^2(5-\lambda).$$

So the last eigenvalue of A is 5.

7. If A is diagonalizable, we have $A = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}P^{-1}$. If neither λ_1 or λ_2 is negative, we can take $B = P\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}P^{-1}$. Then

$$B^2 = \left(P\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}P^{-1}\right)\left(P\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}P^{-1}\right) = P\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}P^{-1} = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}P^{-1} = A.$$

Char poly of A: $|A-\lambda I| = (7-\lambda)(-2-\lambda)-(-18) = \lambda^2-5\lambda+4 = (\lambda-1)(\lambda-4)$.
The eigenvalues of A are distinct, so A is diagonalizable. They are both positive, so we may take B as above.

1-eigenspace:

$$A-I = \begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

An eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

4-eigenspace:

$$A-4I = \begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

An eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So $P=\begin{pmatrix}1&2\\1&1\end{pmatrix}$, $D=\begin{pmatrix}1&0\\0&4\end{pmatrix}$. P^{-1}: $\left(\begin{array}{cc|cc}1&2&1&0\\1&1&0&1\end{array}\right)\to\left(\begin{array}{cc|cc}1&2&1&0\\0&-1&-1&1\end{array}\right)$

$\to\left(\begin{array}{cc|cc}1&2&1&0\\0&1&1&-1\end{array}\right)\to\left(\begin{array}{cc|cc}1&0&-1&2\\0&1&1&-1\end{array}\right)$.

So $A=\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&4\end{pmatrix}\begin{pmatrix}-1&2\\1&-1\end{pmatrix}$, and $B=\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}-1&2\\1&-1\end{pmatrix}=\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}-1&2\\2&-2\end{pmatrix}$

$=\begin{pmatrix}3&-2\\1&0\end{pmatrix}$.

Let's check: $B^2=\begin{pmatrix}3&-2\\1&0\end{pmatrix}\begin{pmatrix}3&-2\\1&0\end{pmatrix}=\begin{pmatrix}7&-6\\3&-2\end{pmatrix}=A$. ✓