Name: Solutions

MIDTERM 1 - JULY 10, 2015

MATH 54 SECTION 8 - ALEX KRUCKMAN

Please put away everything except scratch paper and pencils/pens.
This exam begins at 8:10am and ends at 10am. You have 110 minutes.
Write your answers, *including complete justifications*, in the spaces provided.
If you finish early or have a question, please make your way quietly to the front of the room, taking care not to disturb the other test takers!

Problem	Out of	Score
1	4	
2	6	
3	14	
4	14	
5	10	
6	12	
Total:	60	

Scores on this exam

HW	Out of	Score
1	15	
2	15	
3	15	
4	15	
Total:	60	

Homework scores

Quiz	Out of	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Quiz scores

(1) (4 points) Find all solutions to the following system of linear equations:

$$2x_1 + 3x_2 + x_4 = 1$$
$$x_1 + -x_3 + 4x_4 = 2$$
$$3x_2 + 2x_3 - 7x_4 = 3$$
$$2x_1 + x_3 = 4$$

Row reduce:

$$\left(\begin{array}{cccc|c} 2 & 3 & 0 & 1 & 1 \\ 1 & 0 & -1 & 4 & 2 \\ 0 & 3 & 2 & -7 & 3 \\ 2 & 0 & 1 & 0 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 2 & 3 & 0 & 1 & 1 \\ 0 & -3/2 & -1 & 7/2 & 3/2 \\ 0 & 3 & 2 & -7 & 3 \\ 0 & -3 & 1 & -1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 2 & 3 & 0 & 1 & 1 \\ 0 & -3/2 & -1 & 7/2 & 3/2 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & -3 & 1 & -3 & 3 \end{array}\right)$$

$row_2 \leftarrow row_2 - \frac{1}{2} row_1$
$row_4 \leftarrow row_4 - row_1$

$row_3 \leftarrow row_3 + 2 row_2$

It's already clear from row 3 that the system is inconsistent.

$\boxed{\text{No solutions}}$

(2) (6 points) Let V be a finite-dimensional vector space which is nontrivial ($V \neq \{\mathbf{0}\}$). Then there are infinitely many sets $B = \{\mathbf{v_1}, \ldots, \mathbf{v_d}\}$ such that B is a basis for V. Explain why. [**Hint:** Show that if $\{\mathbf{v_1}, \ldots, \mathbf{v_d}\}$ is a basis, then $\{c\mathbf{v}_1, \ldots, c\mathbf{v_d}\}$ is also a basis for any scalar $c \neq 0$.]

Following the hint, if $\{\bar{v}_1, \ldots, \bar{v}_d\}$ is a basis, so is $\{c\bar{v}_1, \ldots, c\bar{v}_d\}$ when $c \neq 0$.

- Linear Independence:
  If $x_1(c\bar{v}_1) + \ldots + x_d(c\bar{v}_d) = \bar{0}$,
  then $c(x_1\bar{v}_1 + \ldots + x_d\bar{v}_d) = \bar{0}$,
  so $x_1\bar{v}_1 + \ldots + x_d\bar{v}_d = \frac{1}{c}\bar{0} = \bar{0}$.
  But $\{\bar{v}_1, \ldots, \bar{v}_d\}$ is linearly independent,
  so $x_1 = 0, \ldots, x_d = 0$.
- Spanning:
  If $\bar{w} \in V$, since $\{\bar{v}_1, \ldots, \bar{v}_d\}$ spans V,
  we can write $w = x_1\bar{v}_1 + \ldots + x_d\bar{v}_d$.
  Then $w = \frac{x_1}{c}(c\bar{v}_1) + \ldots + \frac{x_d}{c}(c\bar{v}_d)$,
  so $\{c\bar{v}_1, \ldots, c\bar{v}_d\}$ also spans

Alternatively, if $B = \{\bar{v}_1, \ldots, \bar{v}_d\}$, then in B-coordinates,

$$[c\bar{v}_1] = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ [c\bar{v}_2] = \begin{pmatrix} 0 \\ c \\ \vdots \\ 0 \end{pmatrix}, \ldots, [c\bar{v}_d] = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c \end{pmatrix}.$$

The matrix

$$([c\bar{v}_1]_B \ [c\bar{v}_2]_B \cdots [c\bar{v}_d]_B) = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}$$

is invertible, so its columns are a basis for $\mathbb{R}^d \Rightarrow \{c\bar{v}_1, \ldots, c\bar{v}_d\}$ is a basis for V.

Now if V is finite-dimensional and nontrivial, it has <u>some</u> (nonempty) basis $\{\bar{v}_1, \ldots, \bar{v}_d\}$, and there are infinitely many nonzero scalars, so the hint shows there are infinitely many other bases.

(3) Recall that $\mathbb{P}_{\leq 3}$ is the space of all polynomials (with real coefficients, in the variable x) of degree at most 3. Let $Z = \{p(x) \mid p(x) \text{ is in } \mathbb{P}_{\leq 3} \text{ and } p(2) = 0\}$.

(a) (3 points) Show that Z is a subspace of $\mathbb{P}_{\leq 3}$.

- The polynomial $p(x) = 0$ is in Z, since $p(2) = 0$.
- If $p, q \in Z$, then $(p+q)(2) = p(2) + q(2) = 0 + 0 = 0$, so $p+q \in Z$
- If $p \in Z$ and $c \in \mathbb{R}$, then $(cp)(2) = cp(2) = c0 = 0$. so $cp \in Z$.

(b) (4 points) You may assume that $B = \{2 - x, 4 - x^2, 8 - x^3\}$ is a basis for Z. Write the polynomial $x^3 - 3x - 2$ in B-coordinates:

$$[x^3 - 3x - 2]_B = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

We want to write $x^3 - 3x - 2 = a(2-x) + b(4-x^2) + c(8-x^3)$.

Expanding the right, $= 2a + 4b + 8c - ax - bx^2 - cx^3$

So we should take $c = -1$, $b = 0$, $a = 3$ (comparing coefficients)

(c) (4 points) Is there a polynomial $q(x)$ such that $\{2 - x, 4 - x^2, 8 - x^3, q(x)\}$ is a basis for $\mathbb{P}_{\leq 3}$? Why or why not?

Yes. $\{2-x, 4-x^2, 8-x^3\}$ is a linearly independent set in $\mathbb{P}_{\leq 3}$, so it can be expanded to a basis. And $\dim(\mathbb{P}_{\leq 3}) = 4$ (since $\{1, x, x^2, x^3\}$ is a basis), so one extra polynomial suffices. [In fact, we can take any $q(x)$ not in Z, i.e. any q such that $q(2) \neq 0$. $q(x) = 1$ works, for example.]

(d) (3 points) Is $Z' = \{p(x) \mid p(x) \text{ is in } \mathbb{P}_{\leq 3} \text{ and } p(2) = 2\}$ a subspace of $\mathbb{P}_{\leq 3}$? Why or why not?

No. There are three possible reasons:

1) $p(x) = 0$ is not in Z', since $p(2) = 0$, not 2.

2) If $p, q \in Z'$, then $(p+q)(2) = p(2) + q(2) = 2 + 2 = 4$, not 2 so $p + q \notin Z'$

3) If $p \in Z'$, then $(2p)(2) = 2(p(2)) = 2 \cdot 2 = 4$, not 2 so $2p \notin Z'$.

(4) Consider the following 1×1, 2×2, and 3×3 matrices:

$$A = \begin{pmatrix} 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$$

(a) (2 points) Find $\det(A)$.

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(b) (3 points) Find $\det(B)$.

$2 \cdot 4 - 3 \cdot 3 = 8 - 9 = \underline{\underline{-1}}$

(c) (4 points) Find $\det(C)$.

Expanding along the First row,

$$2\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 2(24-25) - 3(18-20) + 4(15-16)$$

$$= 2(-1) - 3(-2) + 4(-1) = -2 + 6 - 4 = \underline{\underline{0}}$$

(d) (3 points) Find $\det(B^{10})$. $[B^{10} = \underbrace{BB \dots B}_{\text{10 times}}]$

$|B^{10}| = |B|^{10} = (-1)^{10} = \underline{\underline{1}}$

(e) (2 points) Find $\det(CC^T)$.

$|CC^T| = |C||C^T| = |C|^2 = \underline{\underline{0}}$

(5) Consider the following matrix:

$$A = \begin{array}{c} \begin{matrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \bar{a}_4 & \bar{a}_5 \end{matrix} \\ \begin{pmatrix} 2 & 4 & 6 & 8 & 10 \\ 2 & 2 & 2 & 10 & 10 \\ 1 & 3 & 5 & 3 & 5 \end{pmatrix} \end{array}$$

(a) (4 points) Is $\mathbf{v} = \begin{pmatrix} 2 \\ -10 \\ 7 \end{pmatrix}$ in the column space of A?

We row reduce to solve the vector equation $x_1\bar{a}_1 + \ldots + x_5\bar{a}_5 = \bar{v}$.

$$\left(\begin{array}{ccccc|c} 2 & 4 & 6 & 8 & 10 & 2 \\ 2 & 2 & 2 & 10 & 10 & -10 \\ 1 & 3 & 5 & 3 & 5 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 2 & 4 & 6 & 8 & 10 & 2 \\ 0 & -2 & -4 & 2 & 0 & -12 \\ 0 & 1 & 2 & -1 & 0 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 2 & 4 & 6 & 8 & 10 & 2 \\ 0 & -2 & -4 & 2 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{row}_2 \leftarrow \text{row}_2 - \text{row}_1$
$\text{row}_3 \leftarrow \text{row}_3 - \frac{1}{2}\text{row}_1$

$\text{row}_3 \leftarrow \text{row}_3 + \frac{1}{2}\text{row}_2$

It is clear from the echelon form that the equation is consistent, so $\bar{v}$ is in $Col(A)$.

(b) (4 points) Find a basis for the null space of A.

Now we want to solve $A\bar{x} = \bar{0}$, so we can pick up row reducing where we left off, but with augmented column $\bar{0}$.

$$\left(\begin{array}{ccccc|c} 2 & 4 & 6 & 8 & 10 & 0 \\ 0 & -2 & -4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 6 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{row}_1 \leftarrow \frac{1}{2}\text{row}_1$
$\text{row}_2 \leftarrow -\frac{1}{2}\text{row}_2$

$\text{row}_1 \leftarrow \text{row}_1 - 2\text{row}_2$

The parametric form of the solution is

$$\bar{x} = \begin{pmatrix} x_3 - 6x_4 - 5x_5 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -6 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

These three vectors are a basis for $Nul(A)$.

(c) (2 points) What is the rank of A?

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(6) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that that $T(\mathbf{e}_1) = 2\mathbf{e}_2$, $T(\mathbf{e}_2) = -\mathbf{e}_3$, and $T(\mathbf{e}_3) = 7\mathbf{e}_1$.

(a) (4 points) Find the standard matrix for T.

$$\left(T(\bar{e}_1)\ T(\bar{e}_2)\ T(\bar{e}_3)\right) = \begin{pmatrix} 0 & 0 & 7 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

(b) (4 points) Explain how you can tell that the matrix in part (a) is invertible. [There are many correct answers]

Here are some possible answers:

- Determinant $= -14 \neq 0$.
- The columns are linearly independent and span $\mathbb{R}^3$
- The matrix row reduces to I (see part (c) below)
- After some row swaps we get

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix} \leftarrow \text{pivot in every row and column.}$$

(c) (4 points) Find the standard matrix for T^{-1}.

$$\underbrace{\left(\begin{array}{ccc|ccc} 0 & 0 & 7 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right)}_{A \qquad I} \to \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 7 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1/7 & 0 & 0 \end{array}\right)_{A^{-1}}$$

$$\boxed{\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & -1 \\ 1/7 & 0 & 0 \end{pmatrix}}$$

Alternatively, note that

$$\begin{aligned} \bar{e}_1 &= T^{-1}(2\bar{e}_2) = 2T^{-1}(\bar{e}_2) & & T^{-1}(\bar{e}_1) = 1/7\,\bar{e}_3 \\ \bar{e}_2 &= T^{-1}(-\bar{e}_3) = -T^{-1}(\bar{e}_3) & \Rightarrow\ & T^{-1}(\bar{e}_2) = 1/2\,\bar{e}_1 \\ \bar{e}_3 &= T^{-1}(7\bar{e}_1) = 7T^{-1}(\bar{e}_1) & & T^{-1}(e_3) = -\bar{e}_2 \end{aligned}$$

columns of

$$\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & -1 \\ 1/7 & 0 & 0 \end{pmatrix}$$