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MATH 54 MIDTERM 1
SUMMER 2011 - SECTION 5 - ALEX KRUCKMAN

Please put away everything except scratch paper and pencils/pens.
You have 110 minutes to complete this exam, which ends at 2pm sharp.
Write your answers, *including complete justifications*, in the spaces provided below.
If you finish early or have a question, please make your way to the front of the room, taking care not to disturb the other test takers!

(1) (8 points) For which values of k does the following system of equations have

(a) No solutions? $k=-1$

(b) Exactly one solution? $k \neq 1$ or -1 $(k^2-1\neq 0)$

(c) Exactly three solutions? Impossible

(d) Infinitely many solutions? $k=1$

$$\begin{cases} x-ky-2z=1 \\ 2x+(1-2k)y+(k-4)z=3 \\ z-x=-2 \end{cases}$$

Row reduce

$$\left(\begin{array}{ccc|c} 1 & -k & -2 & 1 \\ 2 & (1-2k) & (k-4) & 3 \\ -1 & 0 & 1 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -k & -2 & 1 \\ 0 & 1 & k & 1 \\ 0 & -k & -1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -k & -2 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right)$$

inconsistent when $k^2-1=0$ and $k-1\neq 0$
Unique solution when $k^2-1\neq 0$
infinitely many solutions when $k^2-1=0$ and $k-1=0$.

In case (d) (if it occurs), describe *in parametric form* all solution vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

For $k=1$, we have

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right),$$ which tells us

$x = 2+z$
$y = 1-z$.

Solutions have the form

$$\begin{pmatrix} 2+z \\ 1-z \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

Parametric form: $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(2) (8 points) Let V be a vector space. Suppose $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is a linearly independent set of vectors in V.

(a) Is $\{(\bar{v}_1 + \bar{v}_2), (\bar{v}_2 + \bar{v}_3), (\bar{v}_3 + \bar{v}_4), (\bar{v}_4 + \bar{v}_1)\}$ linearly independent? Why or why not?

$(\bar{v}_1 + \bar{v}_2) + -(\bar{v}_2 + \bar{v}_3) + (\bar{v}_3 + \bar{v}_4) + -(\bar{v}_4 + \bar{v}_1) = \bar{0}$,

so these vectors are linearly dependent.

Another way to solve this (other than divine inspiration):

Solve $a(\bar{v}_1 + \bar{v}_2) + b(\bar{v}_2 + \bar{v}_3) + c(\bar{v}_3 + \bar{v}_4) + d(\bar{v}_4 + \bar{v}_1) = \bar{0}$

which is $(a+d)\bar{v}_1 + (a+b)\bar{v}_2 + (b+c)\bar{v}_3 + (c+d)\bar{v}_4 = \bar{0}$.

Since $\{\bar{v}_1, \ldots, \bar{v}_4\}$ are independent, this holds when $\begin{cases} a+d=0 \\ a+b=0 \\ b+c=0 \\ c+d=0 \end{cases}$ only.

This is a system of linear equations, corresponding to the matrix

$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$. Row reducing provides the solution above and others.

(b) Is $\{(\bar{v}_1 + \bar{v}_2), (\bar{v}_2 + \bar{v}_3), (\bar{v}_3 + \bar{v}_1)\}$ linearly independent? Why or why not?

Suppose $a(\bar{v}_1 + \bar{v}_2) + b(\bar{v}_2 + \bar{v}_3) + c(\bar{v}_3 + \bar{v}_1) = \bar{0}$.

Then $(a+c)\bar{v}_1 + (a+b)\bar{v}_2 + (b+c)\bar{v}_3 = \bar{0}$.

Since $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ are independent, $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ are independent,

so we must have $\begin{matrix} a+c=0 \\ a+b=0 \\ b+c=0 \end{matrix}$ and Then $a=-c$, and $a=-b$, so $-b=-c$, but $b+c=0$, $b=c$

so $2b=2c=0$, and b and c are both 0. Then a is 0 also.

We can also look at this as a system of linear equations corresponding to the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. This matrix is invertible (det 2) so the homogeneous system has only the trivial solution $a=b=c=0$.

Hence the vectors are linearly independent.

(3) (8 points) Let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be the function which sums the entries of a vector and returns a vector consisting of three copies of that sum. So

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the function which assigns to a vector the vector consisting of the average of the first two entries and the average of the last two entries. So

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x+y}{2} \\ \frac{y+z}{2} \end{pmatrix}.$$

(a) Show that S is a linear transformation, and find its standard matrix.

Let $\bar{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\bar{v} = \begin{pmatrix} x' \\ y' \end{pmatrix}$. Then $S(\bar{u}+\bar{v}) = S\begin{pmatrix} x+x' \\ y+y' \end{pmatrix} = \begin{pmatrix} x+x'+y+y' \\ x+x'+y+y' \\ x+x'+y+y' \end{pmatrix}$

$= \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix} + \begin{pmatrix} x'+y' \\ x'+y' \\ x'+y' \end{pmatrix} = S\begin{pmatrix} x \\ y \end{pmatrix} + S\begin{pmatrix} x' \\ y' \end{pmatrix} = S(\bar{u}) + S(\bar{v})$.

$S(c\bar{u}) = S\begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} cx+cy \\ cx+cy \\ cx+cy \end{pmatrix} = c\begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix} = cS(\bar{u})$. So S is a linear transformation.

$S(\bar{e}_1) = \begin{pmatrix} 1+0 \\ 1+0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $S(\bar{e}_2) = \begin{pmatrix} 0+1 \\ 0+1 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. So S is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

(b) A similar argument shows that T is a linear transformation - you do not need to show it here! Find the standard matrix for T.

$T(\bar{e}_1) = \begin{pmatrix} \frac{1+0}{2} \\ \frac{0+0}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, $T(\bar{e}_2) = \begin{pmatrix} \frac{0+1}{2} \\ \frac{1+0}{2} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, $T(\bar{e}_3) = \begin{pmatrix} \frac{0+0}{2} \\ \frac{0+1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

So T is $\begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$.

(c) Find the standard matrix for $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$.

$T \circ S$ is $\begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(d) Find the standard matrix for $S \circ T : \mathbb{R}^3 \to \mathbb{R}^3$.

$S \circ T$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 \end{pmatrix}$

(4) **(9 points)** Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 5 & \frac{1}{3} & 0 & 2 \\ 0 & \frac{1}{4} & -1 & 0 & 5 & 0 \\ 0 & 0 & 6 & -\frac{1}{2} & 0 & 2 \\ 0 & 0 & 0 & 3 & 8 & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Compute the determinants:
(a) $|A|$
(b) $|B|$
(c) $|AABA^{-1}BABBBAB^{-1}|$

a) $1 \cdot \frac{1}{4} \cdot 6 \cdot 3 \cdot \frac{1}{9} \cdot 1 = \frac{1}{2}$

b) $-1 \cdot \begin{vmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \frac{1}{2} & 0 \end{vmatrix} = -1 \cdot -2 \cdot \begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 4 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 0 \end{vmatrix} = -1 \cdot -2 \cdot -2 \begin{vmatrix} 0 & 0 & \frac{1}{2} \\ 4 & 1 & 0 \\ 1 & \frac{1}{2} & 0 \end{vmatrix}$

$= -1 \cdot -2 \cdot -2 \cdot \frac{1}{2} \begin{vmatrix} 4 & 1 \\ 1 & \frac{1}{2} \end{vmatrix} = -1 \cdot -2 \cdot -2 \cdot \frac{1}{2} \cdot 1 = -2$

c) $\frac{1}{2} \cdot \frac{1}{2} \cdot -2 \cdot 2 \cdot -2 \cdot \frac{1}{2} \cdot -2 \cdot -2 \cdot -2 \cdot \frac{1}{2} \cdot -\frac{1}{2} = 2$

A A B A^{-1} B A B B B A B^{-1}

(5) (9 points) Let $C^\infty[0,1]$ be the vector space of analytic functions (i.e. functions which have derivatives of all orders) $[0,1] \to \mathbb{R}$. For each the subsets of $C^\infty[0,1]$ specified below, determine whether it is or is not a subspace and explain.

(a) $\{f : f(x) \geq 0 \text{ for all } x \text{ in } [0,1]\}$, that is, nonnegative functions

No. Not closed under scalar multiplication

If f is a nonnegative function ($f(x) \geq 0$ for all x),

$-f$ is a nonpositive function ($f(x) \leq 0$ for all x).

(b) $\{f : f' = 1\}$, that is, functions which differentiate to the constant function 1

No. $\bar{0}$ is not in the subset.

$0' = 0 \neq 1$

(c) $\{f : f'' - f = 0\}$, that is, functions which are equal to their second derivative

Let W be the subset of $C^\infty[0,1]$ in question.

Yes. • $0'' - 0 = 0 - 0 = 0$, so 0 is in W.

• If f and g are in W, $(f+g)'' - (f+g) = f'' + g'' - f - g = (f'' - f) + (g'' - g) = 0$. So $f+g$ is in W.

• If f is in W and c is a scalar, $(cf)'' - cf = c(f'' - f) = c \cdot 0 = 0$. So cf is in W.

(6) (8 points) Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(a) Compute A^{-1}.

Row reduce

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

A^{-1}

(b) Explain why the following set of vectors is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

By the invertible matrix theorem, the colums of A are linearly independent and span $\mathbb{R}^3$.

That is, they are a basis for $\mathbb{R}^3$.