HOMEWORK #10 SOLUTIONS - 8/4/15

Section 4.5 #46: There are three distinct cases to consider:

Case 1: $\lambda = 0$. Then the equation is $y'' = 0$, with general solution $c_1 + c_2 t$, since the only root of the auxiliary polynomial x^2 is 0, with multiplicity 2.

To find a particular solution, we guess $y = A\cos(t) + B\sin(t)$. Then $y' = -A\sin(t) + B\cos(t)$ and $y'' = -A\cos(t) - B\sin(t)$. Plugging in, we have

$$-A\cos(t) - B\sin(t) = \sin(t)$$

so $A = 0$ and $B = -1$.

So we have the general solution $y(t) = c_1 + c_2 t - \sin(t)$. Plugging in the boundary value information, we want $y(0) = c_1 + c_2 \cdot 0 - \sin(0) = 0$, so $c_1 = 0$, and $y(\pi) = c_1 + c_2\pi - \sin(\pi) = 1$, so $c_2\pi = 1$, and $c_2 = 1/\pi$. A solution exists!

Case 2: $\lambda = 1$. Then the equation is $y'' + y = 0$, with general solution $c_1\cos(t) + c_2\sin(t)$, since the auxiliary polynomial is $x^2 + 1 = 0$, with roots $\pm i$.

To find a particular solution, we guess $y = At\cos(t) + Bt\sin(t)$. Note we have to add a power of t, since $\sin(t)$ and $\cos(t)$ are solutions to the homogeneous equation. Then $y' = A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t) = (A + Bt)\cos(t) + (B - At)\sin(t)$, and $y'' = -(A+Bt)\sin(t) + B\cos(t) + (B - At)\cos(t) - A\sin(t) = (2B - At)\cos(t) + (-2A - Bt)\sin(t)$. Plugging in, we have

$$(2B - At)\cos(t) + (-2A - Bt)\sin(t) + (At\cos(t) + Bt\sin(t)) = \sin(t)$$

$$2B\cos(t) - 2A\sin(t) = \sin(t),$$

so $B = 0$ and $A = -\frac{1}{2}$.

So we have the general solution $y(t) = c_1\cos(t) + c_2\sin(t) - \frac{1}{2}t\cos(t)$. Plugging in the boundary value information, we want $y(0) = c_1\cos(0) + c_2\sin(0) - \frac{1}{2}(0)\cos(0) = 0$, so $c_1 = 0$,

and $y(\pi) = c_1\cos(\pi) + c_2\sin(\pi) - \frac{1}{2}(\pi)\cos(\pi) = 1$, so $\frac{\pi}{2} = 1$, which is nonsense. There is no solution!

Case 3: λ neither 0 nor 1. Then the equation is $y'' + \lambda^2 y = 0$, with general solution $c_1\cos(\lambda t) + c_2\sin(\lambda t)$, since the auxiliary polynomial is $x^2 + \lambda^2 = 0$, with roots $\pm\lambda i$.

We guess $y = A\cos(t) + B\sin(t)$. Then $y' = -A\sin(t) + B\cos(t)$ and $y'' = -A\cos(t) - B\sin(t)$. Plugging in, we have

$$(-A\cos(t) - B\sin(t)) + \lambda^2(A\cos(t) + B\sin(t)) = \sin(t),$$

so $-A + \lambda^2 A = 0$ and $-B + \lambda^2 B = 1$. Factoring, $A(\lambda^2 - 1) = 0$ and $B(\lambda^2 - 1) = 1$, so $A = 0$ and $B = 1/(\lambda^2 - 1)$ (this is ok, since $\lambda \neq 1$).

So we have the general solution $y(t) = c_1\cos(\lambda t) + c_2\sin(\lambda t) + \frac{1}{\lambda^2-1}\sin(t)$. Plugging in the boundary value information, we want $y(0) = c_1\cos(0) + c_2\sin(0) + \frac{1}{\lambda^2-1}\sin(0) = 0$, so $c_1 = 0$, and $y(\pi) = c_1\cos(\lambda\pi) + c_2\sin(\lambda\pi) + \frac{1}{\lambda^2-1}\sin(\pi) = 1$, so $c_2\sin(\lambda\pi) = 1$. This has a solution ($c_2 = 1/\sin(\lambda\pi)$) if and only if $\sin(\lambda\pi) \neq 0$, if and only if λ is not an integer.

To sum up: The boundary value problem has a solution when $\lambda = 0$, but not when $\lambda = 1$, and not when λ is any other integer. That is, it has a solution if and only if $\lambda \neq \pm 1, \pm 2, \pm 3, \ldots$.

Section 4.6 #8: The general solution to $y'' + 4y = 0$ is $c_1\cos(2t) + c_2\sin(2t)$.

We set $y = v_1(t)\cos(2t) + v_2(t)\sin(2t)$. Variation of parameters gives the system of equations

$$\begin{cases} v_1'\cos(2t) + v_2'\sin(2t) = 0 \\ v_1'(-2\sin(2t)) + v_2'(2\cos(2t)) = \csc^2(2t) \end{cases}.$$

Writing this as

$$\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \csc^2(2t) \end{pmatrix},$$

we have

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \csc^2(2t) \end{pmatrix}$$

$$= \frac{1}{2(\cos^2(2t) + \sin^2(2t))} \begin{pmatrix} 2\cos(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 0 \\ \csc^2(2t) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -\csc(2t) \\ \cos(2t)\csc^2(2t) \end{pmatrix}$$

So $v_1 = \int -\frac{1}{2}\csc(2t)\,dt = \frac{1}{4}\ln|\csc(2t) + \cot(2t)| + c_1$ and $v_2 = \int \frac{1}{2}\cos(2t)\csc^2(2t)\,dt = -\frac{1}{4}\csc(2t) + c_2$.

General solution: $(\frac{1}{4}\ln|\csc(2t) + \cot(2t)| + c_1)\cos(2t) + (-\frac{1}{4}\csc(2t) + c_2)\sin(2t)$.

Simplifying: $\frac{1}{4}\ln|\csc(2t) + \cot(2t)|\cos(2t) - \frac{1}{4} + c_1\cos(2t) + c_2\sin(2t)$.

Of course, there are other ways of writing the solution, using trig identities.