Name: ____________________

MATH 54 FINAL EXAM
SUMMER 2011 - SECTION 5 - ALEX KRUCKMAN

Please put away everything except scratch paper and pencils/pens.
You have 110 minutes to complete this exam, which ends at 2pm sharp.
Write your answers, *including complete justifications*, in the spaces provided below.
If you finish early or have a question, please make your way to the front of the room, taking care not to disturb the other test takers!

(1) (7 points) Let L be the linear transformation given by $L(y) = 3t^2y'' + 11ty' - 3y$. You may use the facts that $L\left(t^{-1}\right) = -8t^{-1}$, $L\left(t^{\frac{1}{3}}\right) = 0$, and $L(t^{-3}) = 0$

(a) What is $\dim(\ker(L))$?

2 ($L(y)=0$ is a 2nd order linear diff. eq.)

(b) Prove that $\{t^{\frac{1}{3}}, t^{-3}\}$ are linearly independent.

They are both solutions to $L(y)=0$, so the Wronskian gives a test for linear independence.

$W[t^{1/3}, t^{-3}](1) = \begin{vmatrix} 1^{1/3} & 1^{-3} \\ 1/3 \cdot 1^{-2/3} & -3\cdot 1^{-4} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1/3 & -3 \end{vmatrix} = -3 - \frac{1}{3} \neq 0.$

So $\{t^{1/3}, t^{-3}\}$ are linearly independent.

(c) Find the general solution to the differential equation $L(y) = -8t^{-1}$.

t^{-1} is a particular solution.
$t^{1/3}$ and t^{-3} are 2 linearly independent solutions to $L(y)=0$. General solution: $y = t^{-1} + c_1 t^{1/3} + c_2 t^{-3}$.

(d) Solve the initial value problem

$$\begin{cases} 3t^2y'' + 11ty' - 3y = -8t^{-1} \\ y(1) = 2 \\ y'(1) = -4 \end{cases}$$

$y = t^{-1} + c_1 t^{1/3} + c_2 t^{-3}$
$y' = -t^{-2} + 1/3 c_1 t^{-2/3} + -3c_2 t^{-4}$

$2 = y(1) = 1 + c_1 + c_2$
$-4 = y'(1) = -1 + 1/3 c_1 + -3c_2$.

Solve: $\begin{cases} c_1 + c_2 = 1 \\ 1/3 c_1 + -3c_2 = -3. \end{cases}$ $c_1 = 0$, $c_2 = 1$ works.

$\boxed{y = t^{-1} + t^{-3}}$

(e) What is the largest interval on which this solution is guaranteed to be unique?

Interval must contain 1 and avoid 0

initial value point

t^{-1} not defined

The largest such interval is $(0, \infty)$.

(2) (6 points) An $n \times n$ matrix U is called **orthogonal** if its columns form an orthonormal set. An $n \times n$ matrix A is called **orthogonally diagonalizable** if it can be written as $A = UDU^{-1}$, where D is a diagonal matrix and U is an orthogonal matrix.

(a) If U is orthogonal, show that $U^TU = I$.

If $U = (\bar{u}_1 \dots \bar{u}_n)$, $\bar{u}_i \cdot \bar{u}_j = 0$ for $i \neq j$,
and $\bar{u}_i \cdot \bar{u}_i = 1$ for all i, since the columns are
orthonormal ($\|\bar{u}_i\| = \sqrt{\bar{u}_i \cdot \bar{u}_i} = 1$).

So
$$U^TU = \begin{pmatrix} \bar{u}_1^T \\ \vdots \\ \bar{u}_n^T \end{pmatrix} (\bar{u}_1 \dots \bar{u}_n) = \begin{pmatrix} \bar{u}_1 \cdot \bar{u}_1 & \bar{u}_1 \cdot \bar{u}_2 & \cdots & u_1 \cdot \bar{u}_n \\ \bar{u}_2 \cdot \bar{u}_1 & \bar{u}_2 \cdot \bar{u}_2 & \cdots & \bar{u}_2 \cdot \bar{u}_n \\ \vdots & \vdots & & \vdots \\ \bar{u}_n \cdot \bar{u}_1 & \bar{u}_n \cdot \bar{u}_2 & \cdots & \bar{u}_n \cdot \bar{u}_n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

(b) If A is orthogonally diagonalizable, show that A is symmetric, that is, $A^T = A$.

By a), $U^TU = I$, so $U^T = U^{-1}$ (since for square matrices, if $AB = I$, then $A = B^{-1}$).

Then $A = UDU^{-1} = UDU^T$.
$$A^T = (UDU^T)^T = (U^T)^T D^T U^T = UDU^T = A$$
since $U^{TT} = U$ (obviously)
and $D^T = D$ (D is diagonal).

(3) (12 points) For each of the following, give an example or explain why no such example exists.

(a) A 2×5 matrix in reduced echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(b) A 3×3 matrix A with $\mathrm{Nul}(A) = \mathbb{R}^3$.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ This is the only example.

(c) A 3×3 matrix A with $\mathrm{rank}(A) = 1$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) A 2×2 matrix A (with real entries) such that

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}.$$

Impossible. If $A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$, then $|A^2| = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -3$. But $|A^2| = |A|^2$, so $|A|^2 = -3$. Impossible if $|A|$ is real (which it is since A has real entries)

(e) A matrix with characteristic polynomial $\lambda^2 + -4\lambda + 4$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$= (\lambda-2)(\lambda-2)$.

Any matrix with 2 as a repeated eigenvalue will do.

(f) A matrix with real entries which is not diagonalizable over the complex numbers.

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. 0 is the only eigenvalue, but $\dim(\mathrm{Nul}(A - 0I)) = \dim(\mathrm{Nul}(A)) = 1$. $(\mathrm{Nul}(A) = \mathrm{Span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\})$

So A has only one eigenvector.

(4) (8 points) Find the general solutions to the following differential equations:

(a) $y'' - 2y' - 15y = 0$

Aux. poly: $x^2 - 2x - 15 = (x-5)(x+3)$.

$$\boxed{y = c_1 e^{5t} + c_2 e^{-3t}}$$

(b) $y''' + 16y'' + 64y' = 0$

Aux. poly: $x^3 + 16x^2 + 64x = x(x+8)^2$.

$$\boxed{y = c_1 + c_2 e^{-8t} + c_3 t e^{-8t}}$$

(c) $\bar{x}' = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} \bar{x}$ E.values: 2, 3.

2-e.vector: $A - 2I = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. $\bar{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3-e.vector: $A - 3I = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$. $\bar{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\boxed{\bar{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}.$$

(d) $\bar{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \bar{x}$.

E.values: $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4$. $\pm 2i$.

$\alpha = 0, \beta = 2$,

2i-evector:

$A - 2iI = \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2}i \\ 0 & 0 \end{pmatrix}$. $\bar{u} = \begin{pmatrix} -i \\ 2 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}^{\bar{a}} + i\overbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}^{\bar{b}}$.

$\bar{x} = c_1\left(\cos 2t\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \sin 2t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) + c_2\left(\sin 2t\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \cos 2t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$

$$= \boxed{c_1\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} + c_2\begin{pmatrix} -\cos 2t \\ 2\sin 2t \end{pmatrix}}$$

(5) (4 points) Write down the form of the undetermined coefficients guess you would use to solve these differential equations. Please do not solve them.

(a) $y'' - 2y' + 5y = (3t^2 + 1)e^t \cos(2t)$

Aux. poly: $x^2 - 2x + 5$. $1 \pm 2i$ are roots. Extra power of t necessary in the guess.

$$y_p = (At^2 + Bt + C)te^t\cos(2t) + (Dt^2 + Et + F)te^t\sin(2t)$$

(b) $\bar{x}' = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} \bar{x} + \begin{pmatrix} 1 \\ t^4 \end{pmatrix}$

$$\bar{x}_p = t^4\bar{a} + t^3\bar{b} + t^2\bar{c} + t\bar{d} + \bar{e}$$

where $\bar{a}, \ldots, \bar{e}$ are 2-entry undetermined constant vectors.

(6) (7 points) Consider the matrix equation $A\bar{x} = \bar{b}$, where:

$$A = \begin{array}{c} \begin{matrix} \bar a_1 & \bar a_2 & \bar a_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}, \quad \bar{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Note $A \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$,

So Col(A) has basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$.

(2)
(a) Find an orthogonal basis for Col(A).

Gram-Schmidt:

Take $\bar{c}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $W_1 = \text{Span}\{\bar{c}_1\}$.

Take $\bar{c}_2 = \bar{a}_2 - \text{proj}_{W_1}(\bar{a}_2) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \dfrac{\bar{a}_2 \cdot \bar{c}_1}{\bar{c}_1 \cdot \bar{c}_1} \bar{c}_1$

$= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \dfrac{-2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\boxed{\text{orthogonal basis: } \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}}$

(1)
(b) Find an orthogonal basis for Nul(A).

$\text{Nul}(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$ from row reduction above

So $\boxed{\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}}$ is a basis for Nul(A), which is automatically orthogonal (just one vector)

(1)
(c) Compute $\text{proj}_{\text{Col}(A)} \bar{b}$.

$\text{proj}_{\text{Col}(A)} \bar{b} = \dfrac{\bar{b} \cdot \bar{c}_1}{\bar{c}_1 \cdot \bar{c}_1} \bar{c}_1 + \dfrac{\bar{b} \cdot \bar{c}_2}{\bar{c}_2 \cdot \bar{c}_2} \bar{c}_2 = \dfrac{0}{2} \bar{c}_1 + \dfrac{0}{3} \bar{c}_2 = \boxed{\bar{0}}$

(2)
(d) Find a least-squares solution $\hat{x}$ to the system $A\bar{x} = \bar{b}$, and compute the least-squares error (the distance between $A\hat{x}$ and $\bar{b}$).

Solve $A\bar{x} = \text{proj}_{\text{col}(A)} \bar{b} = \bar{0}$. $\boxed{\hat{x} = \bar{0}}$

$\text{dist}(\hat{x}, \bar{b}) = \|\bar{0} - \bar{b}\| = \|\bar{b}\| = \sqrt{1^2 + 1^2 + (-2)^2} = \boxed{\sqrt{6}}$

(1)
(e) Write down, but do not solve, the normal equations for $A\bar{x} = \bar{b}$.

$A^T A = A^T \bar{b}$. $\boxed{\begin{pmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{pmatrix} \bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$

$A^T A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{pmatrix}$, $A^T \bar{b} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(7) (6 points) Find the general solution to the following system of linear differential equations:

$$\begin{cases} x'' = x' + y \\ y' = y \end{cases}$$

Add variables: $x_1 = x$, $x_2 = x'$, $x_3 = y$.

$$\begin{cases} x'' = x' + y \\ y' = y \end{cases} \rightsquigarrow \begin{cases} x_1' = x_2 \\ x_2' = x_2 + x_3 \\ x_3' = x_3 \end{cases} \rightsquigarrow \bar{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \bar{x}$$

Evalues: $0, 1$.

0-eigenvector:

$$A - 0I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad \bar{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad \text{Solution: } e^{0t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

1-eigenvector:

$$A - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad \bar{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad \text{Solution: } e^t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \\ 0 \end{pmatrix}.$$

Generalized 1-eigenvector:

$$(A - I)^2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \mathrm{Nul}((A-I)^2) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\bar{v}}\right\}$$

<u>Solution:</u>

$$e^{At}\bar{v} = e^t e^{(A-I)t}\bar{v} = e^t\left(I\bar{v} + t(A-I)\bar{v} + \tfrac{t^2}{2}(A-I)^2\bar{v} + \dots\right)$$

$$= e^t\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \bar{0} + \bar{0} + \dots\right) = \begin{pmatrix} -e^t \\ 0 \\ e^t \end{pmatrix} + \begin{pmatrix} te^t \\ te^t \\ 0 \end{pmatrix}.$$

General solution:

$$\bar{x} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \\ 0 \end{pmatrix} + c_3\begin{pmatrix} te^t - e^t \\ te^t \\ e^t \end{pmatrix}.$$

<u>Or</u>

$$\boxed{\begin{aligned} x &= c_1 + c_2 e^t + c_3(te^t - e^t) \\ y &= c_3 e^t \end{aligned}}$$