This is a typo. Should be $+25y$.

Problem 1.1. (*2 points*) Find the general solution to the differential equation $y''+10y'+25=0$.

Aux. poly: $x^2+10x+25=(x+5)^2$.

General solution: $y=c_1e^{-5t}+c_2te^{-5t}$.

Problem 1.2. (*2 points*) Prove that your y_1 and y_2 from the previous problem are linearly independent.

$$W[e^{-5t},te^{-5t}](0)=\begin{vmatrix} e^0 & 0e^0 \\ -5e^0 & \cancel{} \end{vmatrix}=\begin{vmatrix} 1 & 0 \\ -5 & 1 \end{vmatrix}=1\neq 0.$$

$(-5\cdot 0\cdot e^0+e^0)$ The Wronskian is nonzero, so the solutions are linearly independent.

Problem 1.3. (*3 points*) Find a particular solution to the equation $y''+10y'+25y=(3t+1)e^t$.

Undetermined Coefficients:

Guess $y=(At+B)e^t$

$y'=Ae^t+(At+B)e^t$

$y''=Ae^t+Ae^t+(At+B)e^t$

$=(2A+B)e^t+Ate^t$

Plug in: $(2A+B)e^t+Ate^t+10((A+B)e^t+Ate^t)+25(At+B)e^t=(3t+1)e^t$

$(12A+36B)e^t+36Ate^t=e^t+3te^t$

$\begin{cases}12A+36B=1\\36A=3.\end{cases}$ $A=1/12$, $12\cdot 1/12+36B=1$, $B=0$. $\boxed{y_p=1/12\,te^t}$

Problem 1.4. (*3 points*) Find the unique solution to the initial-value problem $y''+10y'+25y=(3t+1)e^t$, $y(0)=-1$, $y'(0)=5$.

General solution: $y=1/12\,te^t+c_1e^{-5t}+c_2te^{-5t}$

$-1=y(0)=0+c_1e^0+0=c_1$

$5=y'(0)=\frac{1}{12}\cdot 0e^0+1/12\,e^0+-5c_1e^0+c_2e^0+-5c_2\cdot 0e^0$

$=1/12-5c_1+c_2$

$c_1=-1$, so $5=1/12+5+c_2$, $c_2=-1/12$.

Solution: $y=1/12\,te^t-e^{-5t}-1/12\,te^{-5t}$

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Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Problem 2.1. (*3 points*) Find the eigenvalues of A.

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -\lambda & 1 \end{vmatrix}$$

$$= -\lambda(\lambda^2-1) - (-\lambda-1) + (1+\lambda) = -\lambda^3+3\lambda+2 = -(\lambda+1)^2(\lambda-2)$$

-1 is a root.

$$\begin{array}{r} \lambda^2-\lambda-2 \\ \lambda+1 \overline{)\lambda^3-3\lambda-2} \\ \underline{\lambda^3+\lambda^2} \\ -\lambda^2-3\lambda \\ \underline{-\lambda^2-\lambda} \\ -2\lambda-2 \end{array}$$

E.values: $-1, 2$.

Problem 2.2. (*4 points*) Find a basis for each eigenspace.

2-eigenspace:

$$A-2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

-1-eigenspace:

$$A+I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{x} = \begin{Bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{Bmatrix}. \quad \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Problem 2.3. (*3 points*) Orthogonally diagonalize A (i.e. find the P and the D)

We need an orthogonal basis for the -1-eigenspace.

Take $\bar{b}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\bar{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, and by Graham-Schmidt, $\bar{b}_2 = \bar{v} - \frac{\bar{v}\cdot\bar{b}_1}{\bar{b}_1\cdot\bar{b}_1}\bar{b}_1$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Then $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^3$ of eigenvectors of A.

$D = P^{-1}AP$, where $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & -1 & -1/2 \\ 1 & 1 & -1/2 \\ 1 & 0 & 1 \end{pmatrix}$.

3

Problem 3.1. (*1 point*) For a square matrix A, carefully define the characteristic polynomial $\chi_A(\lambda)$ of A.

$\chi_A(\lambda) = |A - \lambda I|$, where we view the determinant as a polynomial in the variable λ.

Problem 3.2. (*2 points*) What is the characteristic polynomial of A good for?

Its roots are the eigenvalues of A

Problem 3.3. (*3 points*) Compute the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 10 & -7 & 5 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} (1-\lambda) & 2 & 2 \\ 0 & (1-\lambda) & 3 \\ 10 & -7 & (5-\lambda) \end{vmatrix} = (1-\lambda)\begin{vmatrix} (1-\lambda) & 3 \\ -7 & (5-\lambda) \end{vmatrix} + 10\begin{vmatrix} 2 & 2 \\ (1-\lambda) & 3 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 - 6\lambda + 5 + 21) + 10(6 + 2\lambda - 2) = -\lambda^3 + 7\lambda^2 - 22\lambda + 66$$

Problem 3.4. (*6 points*) Given a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and a square matrix A, define $p(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$ (i.e. replace each power of x with the corresponding power of A. Suppose that A is diagonalizable. Show that $\chi_A(A) = 0$ (i.e. when you plug A into its own characteristic polynomial, you get 0). [*Hint: let v be an eigenvector of A. What is $\chi_A(A)v$? How many eigenvectors does A have?*]

A is diagonalizable, so there is a basis of $\mathbb{R}^n$, $\{\bar{v}_1, \ldots, \bar{v}_n\}$ consisting of eigenvectors of A.

Now for any $\bar{v}_i$, $\chi_A(A)\bar{v}_i = (a_n A^n + \ldots + a_1 A + a_0 I)\bar{v}_i$

$= a_n A^n \bar{v}_i + \ldots + a_1 A\bar{v}_i + a_0 I\bar{v}_i = a_n \lambda_i^n \bar{v}_i + \ldots + a_1 \lambda_i \bar{v}_i + a_0 \bar{v}_i$ where λ_i is the e.value of $\bar{v}_i$.

$= (a_n \lambda_i^n + \ldots + a_1 \lambda_i + a_0)\bar{v}_i = \chi_A(\lambda_i)\bar{v}_i = \bar{0}$,

since the e.values of A are roots of its characteristic polynomial.

$\chi_A(A)$ kills each $\bar{v}_i$, and they are a basis for $\mathbb{R}^n$, so $\chi_A(A)$ kills any vector in $\mathbb{R}^n$ – so $\chi_A(A)$ is the 0 matrix.

4 Short answer

(3 points each)

1. Find the form of a particular solution to the differential equation $y'' - 11y' - 42y = t^2 e^{3t}$ (do not solve).

Guess: $y_p = (At^2 + Bt + C)e^{3t}$

2. Determine whether the following vector-valued functions are dependent or independent on $\mathbb{R}$. Justify your answer.

$$x_1(t) = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} e^t \\ e^t \\ 0 \end{pmatrix}, \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

This is tricky - the Wronskian is a test for independence IF the vector functions are all solutions to $\bar{x}' = A\bar{x}$ for same A. These are not. Rather, if

$c_1\bar{x}_1 + c_2\bar{x}_2 + c_3\bar{x}_3 = \bar{0}$, then $\begin{pmatrix} c_1 t + c_2 e^t + c_3 \\ c_1 t + c_2 e^t + c_3 \\ c_1 0 + c_2 0 + c_3 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and in particular,

$c_1 t + c_2 e^t + c_3 = 0$. $\{t, e^t, 1\}$ are independent, so $c_1 = c_2 = c_3 = 0$, and $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are independent.

If you aren't convinced of this, they are all solutions of $y''' - y'' = 0$, and

$W[t, e^t, 1](0) = \begin{vmatrix} t & e^t & 1 \\ 1 & e^t & 0 \\ 0 & e^t & 0 \end{vmatrix}(0)$

$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$

3. Let A be a square matrix, and let λ be a real eigenvalue of A with eigenvector u. Prove that the vector-valued function $x(t) = e^{\lambda t}u$ is a solution to the system $x' = Ax$.

$\bar{x}' = \lambda e^{\lambda t}\bar{u}$

$A\bar{x} = A(e^{\lambda t}\bar{u}) = e^{\lambda t}A\bar{u} = \lambda e^{\lambda t}\bar{u}$.

So $\bar{x}' = A\bar{x}$.

4. Suppose that A is a 4×4 matrix. Suppose that v_1, v_2, v_3 are nonzero vectors in $\mathbb{R}^4$ such that

$$Av_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Av_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \quad Av_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since those are in Col(A)

Determine the minimum and maximum possible rank (i.e. dimension of column space) of A.

$$\dim(\mathrm{Col}(A)) \geq \dim \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}\right\} = 2$$

$$\dim(\mathrm{Nul}(A)) \geq \dim \mathrm{Span}\{\vec{v_3}\} = 1,$$

since this is in Nul(A)

So the rank is at least 2 and at most 3

min. max.

5. Suppose that T is an $n \times n$ matrix satisfying $T^2 = -I_n$. Prove that n must be even.

$$|-I_n| = \begin{vmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{vmatrix} = (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

$$|T^2| = |T|^2 > 0. \text{ So if } T^2 = -I_n, n \text{ is even.}$$

Problem 6.1. (*3 points*) Suppose that a square matrix A satisfies $(A - \lambda I)^k = 0$ for some positive integer k. Prove that the only eigenvalue of A is λ.

Suppose $\bar{v}$ is an eigenvector of A with eigenvalue λ'.
Then $(A-\lambda I)\bar{v} = A\bar{v} - \lambda I\bar{v} = \lambda'\bar{v} - \lambda\bar{v} = (\lambda'-\lambda)\bar{v}$.
So $(A-\lambda I)^k\bar{v} = (\lambda'-\lambda)^k\bar{v} = \bar{0}$ since $(A-\lambda I)^k = 0$.
But $\bar{v} \neq \bar{0}$, so $(\lambda'-\lambda)^k = 0$. Hence $\lambda'-\lambda = 0$, and $\lambda' = \lambda$. This shows that λ is the only eigenvalue of A.

Problem 6.2. (*5 points*) Let $B = \begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{pmatrix}$. Compute e^{tB}.

$$|B-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ -3 & -1-\lambda & 1 \\ 9 & 3 & -4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} (-1-\lambda) & 1 \\ 3 & (-4-\lambda) \end{vmatrix} - \begin{vmatrix} -3 & 1 \\ 9 & (-4-\lambda) \end{vmatrix} - \begin{vmatrix} -3 & (-1-\lambda) \\ 9 & 3 \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2+5\lambda+1) - (3\lambda+3) - (9\lambda)$$

$$= -\lambda^3 - 3\lambda^2 - 3\lambda - 1 = -(\lambda+1)^3. \text{ Eigenvalue: } -1.$$

$$B+I = \begin{pmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{pmatrix} \quad (B+I)^2 = \begin{pmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{pmatrix}\begin{pmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & 0 & 3 \end{pmatrix}$$

$$(B+I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$e^{tB} = e^{-t}e^{t(B+I)} = e^{-t}\left(I + t(B+I) + \tfrac{t^2}{2}(B+I)^2 + \tfrac{t^3}{3!}(B+I)^3 + \ldots\right)$$

$$= \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} + te^{-t}\begin{pmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{pmatrix} + e^{-t}\tfrac{t^2}{2}\begin{pmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & 0 & 3 \end{pmatrix} + 0 + 0 + \ldots$$

$$= \begin{pmatrix} (e^{-t}+3te^{-t}+-\tfrac{3}{2}t^2e^{-t}) & te^{-t} & (-te^{-t}+\tfrac{1}{2}t^2e^{-t}) \\ -3te^{-t} & e^{-t} & te^{-t} \\ (9te^{-t}-\tfrac{9}{2}t^2e^{-t}) & 3te^{-t} & (e^{-t}-3te^{-t}+\tfrac{3}{2}t^2e^{-t}) \end{pmatrix}$$