Math Final Exam Name:____________________

Section 3, Summer 2009 SID:____________________

Instructions/hints

1. Check your own work whenever possible. Although it may involve additional computation, it will be good for your grade.
2. To get partial credit, you will need to show your work clearly. Don't make me try to guess what you're thinking: show me!
3. If you're using a theorem from class or from the textbook, indicate that clearly. You can write something like "by a theorem from class/the textbook..." You don't need to refer to theorems by number, but if a theorem has a name, it would be good to use it.
4. If you get stuck on a problem, don't panic. Ask yourself what all the words in the problem mean. Write down the relevant definitions and see if that helps. If you're still stuck, move on to the next problem and come back later.

For instructor's use only:

Problem		Points	Out of
1			10
2			10
3			12
4			15
5			5
6			8
Bonus			6
Total			60

1

Problem 1.1. (*2 points*) Find the general solution to the differential equation $y''+10y'+25 = 0$.

Problem 1.2. (*2 points*) Prove that your y_1 and y_2 from the previous problem are linearly independent.

Problem 1.3. (*3 points*) Find a particular solution to the equation $y'' + 10y' + 25y = (3t+1)e^t$.

Problem 1.4. (*3 points*) Find the unique solution to the initial-value problem $y'' + 10y' + 25y = (3t+1)e^t,\ y(0) = -1,\ y'(0) = 5$.

2

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Problem 2.1. (*3 points*) Find the eigenvalues of A.

Problem 2.2. (*4 points*) Find a basis for each eigenspace.

Problem 2.3. (*3 points*) Orthogonally diagonalize A (i.e. find the P and the D)

3

Problem 3.1. (*1 point*) For a square matrix A, carefully define the characteristic polynomial $\chi_A(\lambda)$ of A.

Problem 3.2. (*2 points*) What is the characteristic polynomial of A good for?

Problem 3.3. (*3 points*) Compute the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 10 & -7 & 5 \end{pmatrix}$$

Problem 3.4. (*6 points*) Given a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and a square matrix A, define $p(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$ (i.e. replace each power of x with the corresponding power of A. Suppose that A is diagonalizable. Show that $\chi_A(A) = 0$ (i.e. when you plug A into its own characteristic polynomial, you get 0). [*Hint: let v be an eigenvector of A. What is $\chi_A(A)v$? How many eigenvectors does A have?*]

4 Short answer

(*3 points each*)

1. Find the form of a particular solution to the differential equation $y'' - 11y' - 42y = t^2e^{3t}$ (do not solve).

2. Determine whether the following vector-valued functions are dependent or independent on $\mathbb{R}$. Justify your answer.

$$x_1(t) = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} e^t \\ e^t \\ 0 \end{pmatrix}, \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

3. Let A be a square matrix, and let λ be a real eigenvalue of A with eigenvector u. Prove that the vector-valued function $x(t) = e^{\lambda t}u$ is a solution to the system $x' = Ax$.

4. Suppose that A is a 4×4 matrix. Suppose that v_1, v_2, v_3 are nonzero vectors in $\mathbb{R}^4$ such that

$$Av_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Av_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \quad Av_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Determine the minimum and maximum possible rank (i.e. dimension of column space) of A.

5. Suppose that T is an $n \times n$ matrix satisfying $T^2 = -I_n$. Prove that n must be even.

5

Problem 5.1. (*5 points*) Suppose that you had a 2×2 first-order linear differential system $x' = Ax$ and you found a fundamental solution set $\{x_1, x_2\}$ of the form

$$x_1(t) = e^{2t}\left[\cos 3t \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin 3t \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$
$$x_2(t) = e^{2t}\left[\sin 3t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \cos 3t \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$

Determine if solutions to this system spiral in or out, clockwise or counterclockwise. Sketch a trajectory of the system.

6

Problem 6.1. (*3 points*) Suppose that a square matrix A satisfies $(A - \lambda I)^k = 0$ for some positive integer k. Prove that the only eigenvalue of A is λ.

Problem 6.2. (*5 points*) Let $B = \begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{pmatrix}$. Compute e^{tB}.

7 Bonus problem

Problem 7.1. (*3 points*) Compute the Fourier series for the function $f(x) = x^2$ for $-\pi \leq x \leq \pi$.

Problem 7.2. (*3 points*) Assume that the Fourier series you computed above converges to the original function (i.e. assume that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(x) + b_n \sin(x)]$ for $x \in [-\pi, \pi]$). Compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.