Name: Solutions

FINAL EXAM - AUGUST 14, 2015

MATH 54 SECTION 8 - ALEX KRUCKMAN

Please put away everything except scratch paper and pencils/pens.

This exam begins at 8:10am and ends at 10am. You have 110 minutes.

Write your answers, *including complete justifications*, in the spaces provided.

If you finish early or have a question, please make your way quietly to the front of the room, taking care not to disturb the other test takers!

Problem	Out of	Score
1	12	
2	~~10~~ 6	
3	6	
4	6	
5	8	
6	8	
7	10	
Total:	~~60~~ 56	

Potentially useful trig identities:

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

(1) (a) (4 points) Find the general solution to

$$y'' - 2y' + y = 0.$$

Auxiliary polynomial: $x^2 - 2x + 1 = (x-1)^2$

Root: 1, with multiplicity 2.

$y(t) = c_1 e^t + c_2 t e^t$.

(b) (4 points) Solve the initial value problem

$$\begin{cases} y'' - 2y' + y = 0 \\ y(0) = 5 \\ y'(0) = 8 \end{cases}.$$

$y = c_1 e^t + c_2 t e^t$

$y' = c_1 e^t + c_2 (e^t + t e^t)$

$$\left.\begin{aligned} 5 &= y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1 \\ 8 &= y'(0) = c_1 e^0 + c_2 (e^0 + 0 e^0) = c_1 + c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 5 \\ c_2 &= 3 \end{aligned}$$

$y(t) = 5e^t + 3te^t$

(c) (4 points) Now find the general solution to the nonhomogeneous equation

$$y'' - 2y' + y = \frac{e^t}{t}.$$

Variation of parameters:

$y = v_1(t) e^t + v_2(t) t e^t$

$$\begin{cases} v_1' e^t + v_2' t e^t = 0 \\ v_1' e^t + v_2' (e^t + t e^t) = \frac{e^t}{t} \end{cases}$$

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e^t/t \end{pmatrix}$$

$$= \frac{1}{e^{2t} + te^{2t} - te^{2t}} \begin{pmatrix} e^t + te^t & -te^t \\ -e^t & e^t \end{pmatrix} \begin{pmatrix} 0 \\ e^t/t \end{pmatrix}$$

$$= \frac{1}{e^{2t}} \begin{pmatrix} -e^{2t} \\ e^{2t}/t \end{pmatrix} = \begin{pmatrix} -1 \\ 1/t \end{pmatrix} \Rightarrow \begin{aligned} v_1 &= \int -1\,dt = -t + c_1 \\ v_2 &= \int 1/t\,dt = \ln|t| + c_1 \end{aligned}$$

$y = -te^t + \ln|t| te^t + c_1 e^t + c_2 te^t$

[Extra Credit]

(2) (a) (4 points) Use the method of undetermined coefficients to find a particular solution to the nohomogeneous system

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}.$$

The guess $e^{-t}\bar{a}$ doesn't work. Why? The homogeneous part has solutions of the form $e^{-t}\bar{u}$.

Guess: $\bar{x} = e^{-t}\bar{a} + te^{-t}\bar{b} = e^{-t}(\bar{a} + t\bar{b})$.

$\bar{x}' = -e^{-t}(\bar{a} + t\bar{b}) + e^{-t}\bar{b} = e^{-t}(\bar{b} - \bar{a} - t\bar{b})$.

Plugging in, $e^{-t}\begin{pmatrix} b_1 - a_1 - tb_1 \\ b_2 - a_2 - tb_2 \end{pmatrix} = e^{-t}\left(\begin{pmatrix} 3 & 8 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$

$= e^{-t}\begin{pmatrix} 3a_1 + 8a_2 + t(3b_1 + 8b_2) + 1 \\ 2a_1 + 3a_2 + t(2b_1 + 3b_2) + 2 \end{pmatrix}$

Equating coefficients gives the system $\begin{cases} b_1 - a_1 = 3a_1 + 8a_2 + 1 \\ -b_1 = 3b_1 + 8b_2 \\ b_2 - a_2 = 2a_1 + 3a_2 + 2 \\ -b_2 = 2b_1 + 3b_2 \end{cases}$ $\xrightarrow[\text{equation}]{\text{matrix}}$ $\begin{pmatrix} 4 & 8 & -1 & 0 \\ 0 & 0 & 4 & 8 \\ 2 & 4 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$

Row reducing gives $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -5/8 \\ 0 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

a_2 is free, so we can take it to be 0, for example.

Then $a_1 = -5/8$, $a_2 = 0$, $b_1 = -3/2$, $b_2 = 3/4$

$$\boxed{\bar{x}_p = e^{-t}\begin{pmatrix} -5/8 \\ 0 \end{pmatrix} + te^{-t}\begin{pmatrix} -3/2 \\ 3/4 \end{pmatrix}}$$

(Just find the solution to the homogeneous part, if you didn't do the extra credit)

(b) (6 points) Now find the general solution to the same system.

Homogeneous part:

$|A - \lambda I| = (3-\lambda)^2 - 16 = \lambda^2 - 6\lambda + 9 - 16 = (\lambda - 7)(\lambda + 1)$

7-eigenspace:

$A - 7I = \begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$ e.vector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

-1-eigenspace:

$A + I = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ e.vector: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Solutions: $\underbrace{c_1 e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{\text{Full credit for this}} + e^{-t}\begin{pmatrix} -5/8 \\ 0 \end{pmatrix} + te^{-t}\begin{pmatrix} -3/2 \\ 3/4 \end{pmatrix}$

(3) (6 points) We found that the general solution to the heat equation:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0$$

has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}.$$

Let $L = \pi$, and find a (formal) solution to the heat equation, subject to the initial condition $u(x,0) = \cos(2x) - 1$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \stackrel{?}{=} \cos(2x) - 1.$$

We take a Fourier sine series for $\cos(2x-1)$.

$\boxed{\text{Using } \sin(nx)\cos(2x) = \frac{1}{2}\left(\sin(nx-2x) + \sin(nx+2x)\right)}$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\left(\cos(2x) - 1\right)dx$$

$$= \frac{2}{\pi}\left(\int_0^{\pi} \frac{1}{2}\left(\sin((n-2)x) + \sin((n+2)x)\right)dx - \int_0^{\pi} \sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(\frac{1}{2(n-2)}\left(-\cos((n-2)x)\right) + \frac{1}{2(n+2)}\left(-\cos((n+2)x)\right) + \frac{1}{n}\cos(nx)\right)\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{1}{2(n-2)}\left(-(-1)^{n-2} + 1\right) + \frac{1}{2(n+2)}\left(-(-1)^{n+2} + 1\right) + \frac{1}{n}\left((-1)^n - 1\right)\right)$$

It's OK if you didn't do this simplification.

$$= \frac{2}{\pi}\left(1 - (-1)^n\right)\left(\frac{1}{2(n-2)} + \frac{1}{2(n+2)} - \frac{1}{n}\right)$$

$$= \frac{2}{\pi}\left(1 - (-1)^n\right)\left(\frac{n^2 + 2n}{2(n-2)(n+2)n} + \frac{n^2 - 2n}{2(n+2)(n-2)n} - \frac{2(n^2-4)}{2(n+2)(n-2)n}\right)$$

$$= \frac{2}{\pi}\left(1 - (-1)^n\right)\left(\frac{4}{n^3 - 4n}\right)$$

$$\text{So } u(x,t) = \sum_{n=1}^{\infty} \frac{8\left(1 - (-1)^n\right)}{\pi(n^3 - 4n)} \sin(nx) e^{-\beta n^2 t}$$

(4) (6 points) Suppose that $f(x)$ and $g(x)$ are both solutions to an order 3 linear homogeneous differential equation with constant coefficients, and suppose that $f(0) = 1$, $f'(0) = 0$, and $f''(0) = 3$, and furthermore $g(0) = -2$, $g'(0) = 0$, and $g''(0) = -6$. Are f and g linearly independent? Why or why not?

They are linearly dependent - Eval_0 is an isomorphism of the solution space with $\mathbb{R}^3$, so f and g are linearly dependent if and only if $\mathrm{Eval}_0(f)$ and $\mathrm{Eval}_0(g)$ are.

$$\mathrm{Eval}_0(f) = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

linearly dependent.

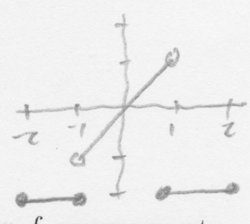
$$\mathrm{Eval}_0(g) = \begin{pmatrix} g(0) \\ g'(0) \\ g''(0) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix}$$

(5) Let $f(x)$ be the function defined on $[-2, 2]$ by

$$f(x) = \begin{cases} -2 & -2 \le x \le -1 \\ x & -1 < x < 1 \\ -2 & 1 \le x \le 2 \end{cases}.$$

(a) (2 points) Does the Fourier series for f consist of only sine terms, only cosine terms, or both?

Both - f is neither even nor odd!



(b) (6 points) To what value does the Fourier series for f converge at

(i) $x = 0$?

0

(ii) $x = 1$?

$$\frac{f(1^-) + f(1^+)}{2} = \frac{1 + (-2)}{2} = -\frac{1}{2}$$

(iii) $x = 2$?

$$\frac{f(-2^+) + f(2^-)}{2} = \frac{(-2) + (-2)}{2} = -2$$

(6) Let A be any $n \times n$ matrix. Recall that a *fundamental matrix* for the equation $\mathbf{x}' = A\mathbf{x}$ is a matrix

$$X(t) = \begin{pmatrix} \mathbf{x_1}(t) & \cdots & \mathbf{x_n}(t) \end{pmatrix}$$

with the property that the columns $\mathbf{x_i}(t)$ are a basis for the space of solutions to the equation.

(a) (4 points) Show that if $X(t)$ and $Y(t)$ are both fundamental matrices for the equation $\mathbf{x}' = A\mathbf{x}$, then there is an $n \times n$ constant matrix C such that $Y(t) = X(t)C$.

Since the columns of Y are solutions, and the columns of X are a basis for the space of solutions, we can write each column $\bar{y}_i$ as $c_1\bar{x}_1 + \dots + c_n\bar{x}_n$ for some constant $c_1, \dots, c_n$.

So $\bar{y}_i = X\bar{c}_i$, where $\bar{c}_i = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

Making these $\bar{c}_i$ the columns of C, we have

$XC = X(\bar{c}_1 \dots \bar{c}_n) = (X\bar{c}_1 \dots X\bar{c}_n) = (\bar{y}_1 \dots \bar{y}_n) = Y$.

(b) (4 points) Assuming the result of part (a), show that if X is any fundamental matrix for $\mathbf{x}' = A\mathbf{x}$, then $e^{At} = X(t)(X(0))^{-1}$.

Since e^{At} is a fundamental matrix for $\bar{x}' = A\bar{x}$, there is a constant matrix C with $e^{At} = X(t)C$. Plugging in $t=0$, $e^{At} = I$,

so $I = X(0)C$. But then $C = X(0)^{-1}$.

So $e^{At} = X(t)\,X(0)^{-1}$.

Aside: Note that we can now use the formula in part (b) to find e^{At} for *any* $n \times n$ matrix A!

(7) In the following PDE, the function $u(x,t)$ is defined for $0 < x < \pi/2$ and $t > 0$:

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \text{ for all } t > 0$$
$$u(\pi/2, t) = 0 \text{ for all } t > 0.$$

(a) (4 points) Assume that a solution has the form $u(x,t) = X(x)T(t)$. Use separation of variables to show that for some constant k, X and T satisfy the ordinary differential equations

$$X''(x) - kX(x) = 0,$$
$$X'(0) = 0, \quad X(\pi/2) = 0$$

and

$$T''(t) + kT(t) = 0.$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t), \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t), \text{ so}$$

$$X(x)T''(t) = -X''(x)T(t)$$

Dividing both sides by $-X(x)T(t)$, $\quad \frac{T''(t)}{-T(t)} = \frac{X''(x)}{X(x)}.$

Both sides must be constant $= K$, so $\quad \frac{T''(t)}{-T(t)} = K = \frac{X''(x)}{X(x)}$

Then $X''(x) = KX(x)$
$\Rightarrow X''(x) - KX(x) = 0$
and $T''(t) = -KT(t)$
$\Rightarrow T''(t) + KT(t) = 0.$

(b) (6 points) Find the values of k such that the equation in X with its boundary condition has nontrivial solutions, and find these solutions. Show all your work (use the back of this page). I will grant you that there are no nontrivial solutions when $K \geq 0$. You may only consider the case $K < 0$.

$X''(x) - KX(x) = 0$ Aux. poly: $x^2 - K$. Since $K<0$, this has complex roots $\pm i\sqrt{-K}$.

General solution: $X = c_1 \cos(\sqrt{-K}\,x) + c_2 \sin(\sqrt{-K}\,x)$
$$X' = -c_1\sqrt{-K}\sin(\sqrt{-K}\,x) + c_2\sqrt{-K}\cos(\sqrt{-K}\,x)$$

Now $X'(0) = c_2\sqrt{-K} = 0 \Rightarrow c_2 = 0$ (since $K \neq 0$).
And $X(\pi/2) = c_1\cos(\sqrt{-K}\,\pi/2) = 0 \Rightarrow c_1 = 0$ (trivial solution) or $\cos(\sqrt{-K}\,\pi/2) = 0$.

The latter occurs when $\sqrt{-K}$ is an odd integer.

So $\sqrt{-K} = n \Rightarrow K = -n^2$ for $n = 1,3,5,7,\ldots$

In this case, the solutions are $X(x) = c_1\cos(nx)$ $(n = 1,3,5,7,\ldots)$

$\boxed{\text{Alternatively, } X(x) = c_1\cos((2n+1)x) \text{ for } n \in \mathbb{N}}$