DIFFERENTIAL EQUATIONS LECTURE NOTES

These notes are intended to supplement sections 6.1 and 4.1 from Nagle, Saff, and Snider. They provide some background and stronger connections to linear algebra which are missing from the text.

INTRODUCTION

An \( n \)th-order differential equation is an equation in the variables \( x, y, y', y'', \ldots, y^{(n)} \).

A solution to a differential equation is a function \( f(x) \) which satisfies the equation when \( f \) is plugged in for \( y \) and the derivatives of \( f \) for \( y', y'', \ldots \).

Examples:

\[
\begin{align*}
y' &= \cos(x) & y'' &= y & xy' &= y \\
\text{Solutions:} & & e^t, e^{-t}, \text{others?} & cx, \text{others?}
\end{align*}
\]

Differential equations usually have multiple solutions. Initial value information can help us narrow it down.

Example: \( y' = \cos(x), y(0) = 3 \).

To solve this initial value problem, we note that \( y = \int \cos(x) \, dx = \sin(x) + c \) describes all solutions to the differential equation. Then \( y(0) = \sin(0) + c = c \), so \( c = 3 \). Hence there is a unique solution: \( y = \sin(x) + 3 \).

General Principle of Differential Equations: (Existence and Uniqueness of Solutions)

Given a differential equation and “enough” initial value information, there should exist a unique solution to the initial value problem.

LINEAR DIFFERENTIAL EQUATIONS

A \( n \)th-order linear differential equation sets a linear combination of \( y, y', \ldots, y^{(n)} \) with coefficients functions of \( x \), equal to a function of \( x \):

\[
a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x).
\]

Dividing by \( a_n \) (this will restrict us to a domain on which \( a_n(x) \neq 0 \)), we get the standard form:

\[
y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = g(x).
\]

The equation is homogeneous if \( g(x) = 0 \).

Theorem (Existence and Uniqueness for Linear Differential Equations)

If \( p_1(x), \ldots, p_n(x) \) and \( g(x) \) are continuous on the interval \((a, b)\), and \( x_0 \) is in \((a, b)\), then for any real numbers \( \gamma_0, \gamma_1, \ldots, \gamma_{n-1} \), the initial value problem

\[
\begin{align*}
y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y &= g(x) \\
y(x_0) &= \gamma_0 \\
y'(x_0) &= \gamma_1 \\
& \vdots \\
y^{(n-1)}(x_0) &= \gamma_{n-1}
\end{align*}
\]

has a unique solution defined on \((a, b)\).
This theorem is proven in more advance courses on differential equations.

**Example:** $y'' = x^{-3}, y(1) = 1, y'(1) = -\frac{1}{2}$.

Note that this is a second order equation, so we need to know two piece of initial value information, $y(x_0)$ and $y'(x_0)$.

If $y'' = x^{-3}$, then integrating, $y' = \int x^{-3} dx = -\frac{1}{2}x^{-2} + c_1$, and integrating again, $y = \frac{1}{2}x^{-1} + c_1x + c_2$. This describes all solutions to the differential equation.

Using the initial value information, $y'(1) = -\frac{1}{2} \cdot 1 + c_1 = -\frac{1}{2} + c_1$, so $c_1 = 0$.
And $y(1) = \frac{1}{2} \cdot 1 + 0 \cdot 1 + c_2 = \frac{1}{2} + c_2$, so $c_2 = \frac{1}{2}$.

We have found a solution: $y = \frac{1}{2}x^{-1} + \frac{1}{2}$. 

But what interval is it defined on? In order to use the theorem to conclude that our solution is unique, we must specify an interval $(a,b)$. It must contain 1 (our $x_0$), and it must not contain 0 (since our $g(x) = x^{-3}$ is not continuous there).

We may as well take the largest interval satisfying these conditions: $(0, \infty)$. By the theorem, $y = \frac{1}{2}x^{-1} + \frac{1}{2}$ is the unique solution to the initial value problem on $(0, \infty)$.

As an example of why we need to be careful about the domain interval, consider the function

$$f(x) = \begin{cases} \frac{1}{2}x^{-1} + \frac{1}{2} & x > 0 \\ \frac{1}{2}x^{-1} - 1 & x < 0 \end{cases}$$

which is defined on $\mathbb{R} \setminus \{0\}$. At every point other than 0, $f$ satisfies $f''(x) = x^{-3}$, and we have $f(1) = 1$ and $f'(1) = -\frac{1}{2}$. That is, $f$ is a solution to the initial value problem. The moral is that uniqueness of the solution may fail if we consider functions defined on larger domains.

**Example:** $xy'' + \sqrt{x+1}y' + \frac{1}{x-1}y = 0, y(x_0) = 0, y'(x_0) = 0$.

Standard form:

$$y'' + \frac{\sqrt{x+1}}{x}y' + \frac{1}{x(x-1)} y = 0.$$ 

All coefficient functions are defined and continuous on $(-1,0)$, $(0,1)$, and $(1,\infty)$. So the theorem guarantees a unique solution defined on

$$\begin{cases} (-1,0) & \text{if } -1 < x_0 < 0 \\ (0,1) & \text{if } 0 < x_0 < 1 \\ (1,\infty) & \text{if } 1 < x_0 \end{cases}.$$ 

Note that if we were given initial value information at three points, one in each interval, then we would get a unique solution on each interval, which we could piece together into a unique solution on the union of the domains.

**LINEAR ALGEBRA PERSPECTIVE**

We are working in a large vector space of functions. This could be

- $C^\infty$, infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$, or
- $C^\infty(a,b)$, infinitely differentiable functions $(a,b) \to \mathbb{R}$, or maybe
- $C^n(a,b)$, $n$ times differentiable functions $(a,b) \to \mathbb{R}$, if we are working with at most $n$th-order differential equations.

For convenience, I will usually use $C^\infty(a,b)$ to denote our function space.
Given an \( n \)th-order linear homogeneous differential equation \( y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \), we may pick an interval \((a, b)\), on which \( p_1, \ldots, p_n \) are defined and continuous. The differential equation then defines a linear transformation \( L : C^\infty(a, b) \to C^\infty(a, b) \), given by
\[
L(f) = f^{(n)} + p_1 f^{(n-1)} + \cdots p_n f.
\]

The solutions to the differential equations are the functions \( f \) in \( C^\infty(a, b) \) such that \( L(f) = 0 \), that is, the kernel of \( L \). Recall that the kernel of a linear transformation is always a subspace of the domain, so \( \ker(L) \) is a subspace of \( C^\infty(a, b) \). We are interested in describing this subspace, which I will call \( S = \ker(L) \).

For any point \( x_0 \) in \((a, b)\) and \( f \) in \( S \), let
\[
\text{Eval}_{x_0}(f) = \begin{pmatrix}
  f(x_0) \\
  f'(x_0) \\
  \vdots \\
  f^{(n-1)}(x_0)
\end{pmatrix}.
\]
This is a vector of \( n \) real numbers. \( \text{Eval}_{x_0} \) is a linear transformation \( S \to \mathbb{R}^n \).

Now a set of initial values \( y(x_0) = \gamma_0, \ldots, y^{(n-1)}(x_0) = \gamma_{n-1} \) is just a target vector in \( \mathbb{R}^n \):
\[
\gamma = \begin{pmatrix}
  \gamma_0 \\
  \gamma_1 \\
  \vdots \\
  \gamma_{n-1}
\end{pmatrix}.
\]
That is, we require \( \text{Eval}_{x_0}(y) = \gamma \). The existence and uniqueness theorem says that for any target vector \( \gamma \) in \( \mathbb{R}^n \), there exists a unique \( f \) in \( S \) such that \( \text{Eval}_{x_0}(f) = \gamma \). The existence says that \( \text{Eval}_{x_0} \) is onto. The uniqueness says that \( \text{Eval}_{x_0} \) is one-to-one.

So \( \text{Eval}_{x_0} \) is an isomorphism \( S \cong \mathbb{R}^n \). This tells us several things:

- \( \dim(S) = n \).
- If we can find \( n \) linearly independent solution functions \( f_1, \ldots, f_n \) in \( S \), they will be a basis for \( S \). That is, all solutions to the differential equation can be uniquely expressed as \( c_1 f_1 + \cdots c_n f_n \) for some real numbers \( c_1, \ldots, c_n \).
- Given initial value information at \( x_0 \), \( y(x_0) = \gamma_0, \ldots, y^{(n-1)}(x_0) = \gamma_{n-1} \), the unique solution to the initial value problem will be \( c_1 f_1 + \cdots + c_n f_n \), where the \( c \)'s are chosen so that \( c_1 \text{Eval}_{x_0}(f_1) + \cdots + c_n \text{Eval}_{x_0}(f_n) = \gamma \).
- Given functions \( f_1, \ldots, f_n \) in \( S \) (that is, solutions to the same differential equation), we can test their linear independence by testing the linear independence of \( \text{Eval}_{x_0}(f_1), \ldots, \text{Eval}_{x_0}(f_n) \). One way to do this is by taking the determinant
\[
W[f_1, \ldots, f_n](x_0) = \begin{vmatrix}
  f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\
  f'_1(x_0) & f'_2(x_0) & \cdots & f'_n(x_0) \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(n-1)}_1(x_0) & f^{(n-1)}_2(x_0) & \cdots & f^{(n-1)}_n(x_0)
\end{vmatrix}.
\]
This determinant is called the Wronskian of \( f_1, \ldots, f_n \), evaluated at \( x_0 \).