CRAMER'S RULE

For the following problems, we define:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 2 & 1 & 2 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Additionally, given a matrix B and a vector $\mathbf{v}$, we define $B_i(\mathbf{v})$ to be the matrix obtained from B by replacing column i with $\mathbf{v}$.

(1) Preliminaries:
   (a) Compute $|A|$.
   (b) Write down $I_2(\mathbf{x})$.
   (c) What is $|I_2(\mathbf{x})|$?

$$|A| = \begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2$$

$$I_2(\bar{x}) = \begin{pmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{pmatrix}$$

$$|I_2(\bar{x})| = x_2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = x_2$$

(2) Suppose $\mathbf{x}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$. Compute the matrix product $A(I_2(\mathbf{x}))$. Can you express this product using our new notation?

$$A I_2(\bar{x}) = \begin{pmatrix} 1 & x_1+2x_2+3x_3 & 3 \\ 0 & 4x_2+6x_3 & 6 \\ 2 & 2x_1+x_2+2x_3 & 2 \end{pmatrix} = (\bar{a}_1 \ A\bar{x} \ \bar{a}_3) = (\bar{a}_1 \ \bar{b} \ a_3) = A_2(\bar{b})$$

(3) Now take determinant of the product from the last question. Can you use the multiplication rule for determinants to find a formula for the unknown x_2 in terms of this determinant and $|A|$?

$$|A_2(\bar{b})| = |A I_2(\bar{x})| = |A||I_2(\bar{x})| = |A| x_2, \text{ so } x_2 = |A_2(\bar{b})|/|A|$$

$$|A_2(\bar{b})| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 0 & 6 \\ 2 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} = -6, \text{ so } x_2 = -3.$$

(4) The method by which you found x_2 is called Cramer's Rule. Use Cramer's Rule to find the other entries of $\mathbf{x}$, solving the equation $A\mathbf{x} = \mathbf{b}$.

$$x_1 = |A_1(\bar{b})|/|A| = \begin{vmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 1 & 1 & 2 \end{vmatrix}/2 = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}/2 = 0, \quad x_3 = |A_3(\bar{b})|/|A| = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & 1 & 1 \end{vmatrix}/2 = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}/2 = 2$$

$$\bar{x} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$$

Extra Problems (if time permits):

Note that if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are solutions to the equations $A\mathbf{u} = \mathbf{e}_1$, $A\mathbf{v} = \mathbf{e}_2$, $A\mathbf{w} = \mathbf{e}_3$, then combining these vectors into the matrix $\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$, we have

$$A\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} = \begin{pmatrix} A\mathbf{u} & A\mathbf{v} & A\mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} = I$$

(1) Use Cramer's Rule to solve the equations $A\mathbf{u} = \mathbf{e}_1$, $A\mathbf{v} = \mathbf{e}_2$, $A\mathbf{w} = \mathbf{e}_3$ (for A as above). Put your answers together to find A^{-1}.

$$u_1 = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 1 & 2 \end{vmatrix}/2 = \begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix}/2 = 1, \quad u_2 = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 0 & 6 \\ 2 & 0 & 6 \end{vmatrix}/2 = -\begin{vmatrix} 0 & 6 \\ 2 & 6 \end{vmatrix}/2 = 6, \quad u_3 = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & 1 & 0 \end{vmatrix}/2 = \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix}/2 = -4$$

$$\bar{u} = \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix}$$

(2) Can you express the entries of A^{-1} in terms of $|A|$ and the cofactors C_{ij} of A? *Hint:* What is $|A_i(\mathbf{e}_j)|$? Compute the determinant by expanding down the i^{th} column.

$$v_1 = \begin{vmatrix} 0 & 2 & 3 \\ 1 & 4 & 6 \\ 0 & 1 & 2 \end{vmatrix}/2 = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}/2 = -1/2, \quad v_2 = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 2 & 0 & 2 \end{vmatrix}/2 = \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}/2 = -2$$

$$v_3 = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 1 \\ 2 & 1 & 0 \end{vmatrix}/2 = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}/2 = 3/2, \quad \bar{v} = \begin{pmatrix} -1/2 \\ -2 \\ 3/2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 6 & -2 & -3 \\ -4 & 3/2 & 2 \end{pmatrix}, \quad |A_i(\bar{e}_j)| = C_{ji}, \text{ so } A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$