Notes on generalized Fraïssé theory

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The purpose of these notes is to give an exposition of Fraïssé theory in an extremely general setting. That is, we drop the hereditary property, instead working with a class of finite structures together with a distinguished class of “strong embeddings” between them. We also weaken the amalgamation property, instead developing the theory with the weak amalgamation property introduced by Kechris and Rosendal (discovered independently under the name “almost amalgamation property” by Ivanov), and sometimes we drop amalgamation altogether, obtaining a generic theory (but not a generic model) under only the assumption of the joint embedding property.

Some may object that the setting is not general enough: we work with finite structures in relational languages throughout (though the setting allows us to consider only structures in which a particular relation is interpreted as the graph of a function), and we do not take a purely categorical approach. All I can say to this is that the theory developed here is the one that seemed the most natural to me at the time of writing.

In Section 1, we give the main definitions and identify the minimal assumption on a strong embedding class which will be necessary for a coherent theory (extendibility). We also discuss a connection with abstract elementary classes and set the descriptive set theory stage for talking about “genericity” later on. Section 2 contains the basic Fraïssé theory. In Section 3, we take up the model theoretic properties of the generic limit (i.e. the Fraïssé limit). Finally, Section 4 addresses the question of pseudofiniteness of the generic limit, through the lens of “robustness” properties à la Macpherson, Steinhorn, and Hill.

Many of the theorems here are direct generalizations of results (classical or more recent) appearing in the literature, though I believe that the choices in presentation are rather unique. I am not aware of prior appearances of the notion of extendibility described in Section 1.2, though it seems quite natural. It is possible that it is known to category theorists, in the context of ind-categories. Example 2.4, which demonstrates that a class can have the weak amalgamation property without containing a cofinal subclass with the amalgamation property, is totally original, as is Section 2.4. The results in Section 4 are new, though they are parallel to and inspired by work of Cameron Hill.

There is much to add, and there are “TO DO” notes scattered throughout the document. The section which needs the most work is Section 3, which proports to be about model-theoretic properties of the generic limit, but which currently only addresses $\aleph_0$-categoricity. A companion section which is currently unwritten is Section 1.4, on definability properties, which will be necessary to get a handle on the model theoretic properties to be discussed in Section 3.
1 Strong embedding classes and their direct limits

1.1 Strong embedding classes

Throughout, we fix a relational language $L$.

**Definition 1.1.** A strong embedding class $K = (S_K, E_K)$ is a class $S_K$ of finite $L$-structures, closed under isomorphism, together with a class $E_K$ of embeddings, called strong embeddings, between members of $S_K$, such that $E_K$ is closed under composition and every isomorphism between members of $S_K$ is in $E_K$.

Let $C_L$ be the category whose objects are $L$-structures and whose arrows are embeddings.

In categorical language, Definition 1.1 says that the objects $S_K$ and arrows $E_K$ form a subcategory of $C_L$ which is full with respect to isomorphisms. We call this subcategory $C_K$.

An immediate consequence of Definition 1.1 is that the class $E_K$ of strong embeddings is closed under isomorphism. That is, if the following diagram in $C_L$ is commutative, then $f$ is strong if and only if $g$ is strong.

\[
\begin{array}{ccc}
B & \cong & B' \\
\downarrow^f & & \downarrow^g \\
A & \cong & A'
\end{array}
\]

**Notation 1.2.** If $A$ is a substructure of $B$, and the inclusion $A \to B$ is a strong embedding, we write $A \preceq_K B$ and say that $A$ is a strong substructure of $B$.

Sometimes we will have occasion to pare down a strong embedding class to a subclass.

**Definition 1.3.** Let $K = (S_K, E_K)$ and $K' = (S_{K'}, E_{K'})$ be strong embedding classes.

1. $K'$ is a subclass of $K$ if $S_{K'} \subseteq S_K$ and $E_{K'} \subseteq E_K$, i.e. $C_{K'}$ is a subcategory of $C_K$.

2. $K'$ is a full subclass of $K$ if additionally when $A, B \in S_{K'}$ and $f : A \to B$ is in $E_K$, then $f \in E_{K'}$. That is, $C_{K'}$ is a full subcategory of $C_K$.

3. $K'$ is a cofinal subclass of $K$ if $K'$ is a full subclass of $K$ and for all $A$ in $K$ there exists $B$ in $K'$ and a strong embedding $A \to B$ in $E_K$.

**Definition 1.4.** A strong embedding class $K$ is a chain class if there is a chain

\[ A_0 \preceq_K A_1 \preceq_K A_2 \preceq_K \ldots \]

such that every structure in $K$ is isomorphic to $A_i$ for some $i$. Note that we do not require that every strong embedding is isomorphic to one of the inclusions $A_i \to A_j$ for $i < j$.

**Example 1.5.** Let $K$ be the class of all finite acyclic graphs of degree at most two (each vertex has at most two neighbors). We will call all embeddings between members of $K$ strong. Let $K'$ be the class of all finite acyclic connected graphs of degree at most two, again equipped with all embeddings between members of $K'$.

$K'$ is a cofinal subclass of $K$, since we can always add new vertices and edges connecting the components of a graph in $K$. Moreover, $K'$ is a chain class, since for all $n$ there is exactly one structure $A_n$ in $K'$ of size $n$ up to isomorphism, and $A_n$ embeds in $A_{n+1}$.
1.2 $K$-direct limits

**Definition 1.6.** We say a structure $M$ is a $K$-direct limit if it is isomorphic to the direct limit in $\mathcal{C}_L$ of some nonempty directed system in $\mathcal{C}_K$. In particular, every structure in $K$ is a $K$-direct limit, and every finite $K$-direct limit is in $K$. If $M \cong \lim_i(A_i)$, we usually identify each $A_i$ with its image in $M$ under the direct limit embedding and write $\pi^M_i$ for the inclusion $A_i \rightarrow M$.

The underlying sets of direct limits in the category of $L$-structures are computed just as in the category of sets. In particular, every element of the direct limit $M \cong \lim_i(A_i)$ is contained in some $A_i$. Generalizing a bit, we have the following basic fact:

**Proposition 1.7.** If $M \cong \lim_i(A_i)$, then for any $A_i$ and any finite subset $B$ of $M$, there is an $A_j$ such that $A_i \subseteq_K A_j$ and $B \subseteq A_j$.

**Proof.** Given $A_i$ and $B \subseteq M$, for each $b \in B$, pick $A_{i_b}$ in the directed system so that $b$ is contained in $A_{i_b}$. Then by directedness, we can find an $A_j$ such that all of the structures $\{A_i\} \cup \{A_{i_b} \mid b \in B\}$ are strong substructures of $A_j$ which does what we want. \qed

We now extend our notion of strong embedding to $K$-direct limits in the natural way, taking the inclusions $\pi^M_i$ to be strong and closing under composition and the universal property of the direct limit. In categorical language, we take the category $\mathcal{C}_{K\infty}$ of $K$-direct limits and strong embeddings to be equivalent to the category ind-$\mathcal{C}_K$ of formal direct limits in $K$.

**Definition 1.8.** Let $M$ and $N$ be $K$-direct limits, given together with direct limit presentations, $M \cong \lim_i(A_i)$ and $N \cong \lim_j(B_j)$, and let $f: M \rightarrow N$ be an embedding. Then $f$ is a strong embedding if for all $A_i$, $f \circ \pi^M_i$ factors as $\pi^N_j \circ g$ for some $B_j$ and some strong embedding $g: A_i \rightarrow B_j$.

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\pi^M_i & & \pi^N_j \\
A_i & \xrightarrow{g} & B_j
\end{array}
$$

Definition 1.8 implies that the class of strong embeddings between $K$-direct limits is closed under composition. Indeed, given $M \cong \lim(A_i)$, $M' \cong \lim(B_j)$, and $M'' \cong \lim(C_k)$, strong embeddings $f: M \rightarrow M'$ and $f': M' \rightarrow M''$, and any $A_i$, the map $(f' \circ f) \circ \pi^M_i$ first factors as $f' \circ \pi^M_j \circ g$ for some $B_j$ and strong $g: A_i \rightarrow B_j$ since $f$ is strong, and then as $\pi^M_k \circ (g' \circ g)$ for some $C_k$ and strong $g': B_j \rightarrow C_k$ since $f'$ is strong. The map $g' \circ g$ is strong since $E_K$ is closed under composition.

$$
\begin{array}{ccc}
M & \xrightarrow{f} & M' & \xrightarrow{f'} & M'' \\
\pi^M_i & & \pi^M_j & & \pi^M_k \\
A_i & \xrightarrow{g} & B_j & \xrightarrow{g'} & C_k
\end{array}
$$
However, the class of strong embeddings may not contain all isomorphisms (as $L$-structures) of $K$-direct limits. To put it another way, whether an embedding $M \to N$ is strong may depend on the presentation of $M$ and $N$ as $K$-direct limits. We rule out this undesirable situation with a definition.

**Definition 1.9.** A strong embedding class $K$ is *extendible* if every isomorphism between $K$-direct limits is strong.

**Notation 1.10.** Let $K$ be an extendible class. We write $K_\infty = (S_{K_\infty}, E_{K_\infty})$, where $S_{K_\infty}$ is the class of structures isomorphic to $K$-direct limits and $E_{K_\infty}$ is the class of strong embeddings between them, and we write $C_{K_\infty}$ for the category of $K$-direct limits and strong embeddings. If $M$ is a substructure of $N$, both are $K$-direct limits, and the inclusion $M \to N$ is a strong embedding, we write $M \preceq_K N$.

**Remark 1.11.** As a special case of Definition 1.8, if $A$ is in $K$ and $A \subseteq N \cong \varprojlim_{i} B_i$, we have $A \preceq_K N$ if and only if $A \preceq_K B_j$ for some $j$.

It is useful to have a more concrete criterion for extendibility, as in the following Proposition.

**Proposition 1.12.** A strong embedding class $K$ is extendible if and only if whenever a “ladder diagram” of the following form, with the $f_i$ and $g_i$ strong embeddings, but the $\alpha_i$ and $\beta_i$ arbitrary embeddings, commutes in $C_L$,

$$
\begin{array}{cccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \cdots \\
\xrightarrow{\alpha_0} \downarrow \beta_0 & & \xrightarrow{\alpha_1} \downarrow \beta_1 & & \xrightarrow{\alpha_2} \downarrow \beta_2 & \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \cdots 
\end{array}
$$

then, for some $n$, the unique map $A_0 \to B_n$ coming from the diagram is a strong embedding.

**Proof.** Suppose $K$ is extendible. Given a ladder diagram, let $M \cong \varprojlim(A_i)$ and $N \cong \varprojlim(B_i)$. Then the $\alpha_i$ induce a map $\alpha: M \to N$, and the $\beta_i$ induce its inverse map $\beta: N \to M$, so $\alpha$ is an isomorphism, and hence a strong embedding. Then the map $\alpha \circ \pi_0^M: A_0 \to N$ must factor through $B_n$ for some $n$ via a strong embedding $h: A_0 \to B_n$, which is the map coming from the diagram.

Conversely, suppose the condition on ladder diagrams is satisfied, and let $\varphi: M \to N$ be an isomorphism of $K$-direct limits, $M \cong \varprojlim(A_i)$ and $N \cong \varprojlim(B_i)$. We would like to show that $\varphi$ is strong, so let $A_{i_0}$ be among the $A_i$. Now the image of $A_{i_0}$ under $\varphi$ is a finite subset of $N$, so by Proposition 1.7, we can find $B_{j_0}$ such that $\varphi[A_{i_0}] \subseteq B_{j_0}$. Let $\alpha_0 = (\varphi \restriction A_{i_0}): A_{i_0} \to B_{j_0}$. This is an embedding, which is not necessarily strong.

Similarly, the image of $B_{j_0}$ under $\varphi^{-1}$ is a finite subset of $M$, so by Proposition 1.7, we can find $A_{i_1}$ such that $A_{i_0} \preceq_K A_{i_1}$ and $\varphi^{-1}[B_{j_0}] \subseteq A_{i_1}$. Let $f_0$ be the strong inclusion $A_{i_0} \preceq A_{i_1}$, and let $\beta_0 = \varphi^{-1} \restriction B_{j_0}: B_{j_0} \to A_{i_1}$. Continuing in this way, we build a ladder diagram, and we conclude that for some $n$, $A_{i_0}$ embeds strongly in $B_n$ via $\varphi$, i.e. the map $\varphi \circ \pi_{i_0}^M$ factors as $\pi_{i_n}^N \circ h_n$, where $h_n = \varphi \restriction A_{i_0}$ is strong.

\[\Box\]
This condition on the class $K$ is still a bit unwieldy. We now give a finitary condition on $K$, smoothness, which is simple to verify and implies extendibility. We also define a weaker notion, coherence.

**Definition 1.13.** Let $K$ be a strong embedding class.

$$
\begin{array}{c}
A \\
\downarrow f \\
\downarrow g
\end{array}
\begin{array}{c}
gof \\
\downarrow C
\end{array}
\begin{array}{c}
B
\end{array}
$$

- $K$ is *smooth* if for all $A, B, C$ in $K$ and $f: A \rightarrow B$ and $g: B \rightarrow C$ embeddings such that $g \circ f$ is strong, then $f$ is also strong.

- $K$ is *coherent* if given $A, B, C,$ in $K$ and $f: A \rightarrow B$ and $g: B \rightarrow C$ embeddings such that $g$ is strong and $g \circ f$ is strong, then $f$ is also strong.

We borrow the term smooth from Kueker and Laskowski, who use it for a slightly stronger condition. In their definition, a class $K$ is smooth if for all $A \in K$, there is a universal theory $T_A$ with parameters from $A$ such that an embedding $f: A \rightarrow B$ is strong if and only if $B \models T_A$ (with the parameters from $A$ interpreted as their images under $f$). Note that this implies our meaning of smooth, since the truth of universal sentences is preserved under substructure.

The sort of definability condition on the class of strong embeddings assumed by Kueker and Laskowski is sometimes useful (see Section 1.4), but to maintain generality, we will assume it only when necessary. But note that if the language $L$ is finite, then any smooth class in our sense is also smooth in the sense of Kueker and Laskowski. Take $T_A$ to consist of the following axioms: for every $B$ in $K$ such that $A$ is a substructure of $B$ but the inclusion is not strong,

$$\forall \overline{b} \neg \varphi_B(\overline{a}, \overline{b}),$$

where $\pi$ enumerates $A$, $\overline{b}$ enumerates $B \setminus A$, and $\varphi_B$ is the conjunction of the diagram of $B$.

Smoothness and coherence extend from $K$ to $K_\infty$.

**Proposition 1.14.** Let $K$ be a strong embedding class. If $K$ is smooth, then so is $K_\infty$. If $K$ is coherent, then so is $K_\infty$.

**Proof.** Suppose we have embeddings of $K$-direct limits $f: M_1 \rightarrow M_2$ and $g: M_2 \rightarrow M_3$. For convenience, we identify $M_1$ and $M_2$ with their images in $M_3$. Further, we assume that $g \circ f$ is strong, so that $M_1 \subseteq M_2$, $M_2 \subseteq M_3$, and $M_1 \preceq_K M_3$. Let $M_1 \cong \lim_{\rightarrow}(A_i)$, $M_2 \cong \lim_{\rightarrow}(B_j)$, $M_3 \cong \lim_{\rightarrow}(C_k)$.

For smoothness, to show that $M_1 \preceq_K M_2$, it suffices to show that for any $A_i$, there is a $B_j$ such that $A_i \preceq_K B_j$. Choose any $B_j$ such that $A_i \subseteq B_j$. Since $M_1 \preceq_K M_3$, there is some $C_k$ such that $A_i \preceq_K C_k$, and we may choose $C_k$ so that $B_j \subseteq C_k$. Now applying smoothness in $K$, we have $A_i \preceq_K B_j$.

For coherence, we further assume that $M_2 \preceq_K M_3$. Again, we need to show that for any $A_i$, there is a $B_j$ such that $A_i \preceq_K B_j$, and we choose any $B_j$ such that $A_i \subseteq B_j$. Now there is some $C_k$ such that $A_i \preceq_K C_k$ and some $C'_k$ such that $B_j \preceq_K C'_k$. By directedness of the
(C_k) system, there is some C_k'' such that A_i ≤_K C_k ≤_K C_k'' and B_j ≤_K C_k'' ≤_K C_k'. Now applying coherence in K, we have A_i ≤_K B_j.

Smoothness implies coherence and extendibility, but the latter conditions are independent.

**Proposition 1.15.** Every smooth class is coherent and extendible.

**Proof.** It is clear from the definition that smoothness implies coherence. For extendibility, we use the characterization of Proposition 1.12. Given a ladder diagram, smoothness implies that since the map f_0 is strong, then already the map α_0: A_0 → B_0 is strong.

**Example 1.16 (A class which is extendible but not coherent).** Let L = {E}, and let K be the class of finite structures in which E is interpreted as an equivalence relation. Say an embedding A → B is strong when it is an isomorphism, or when B contains more equivalence classes than A.

K is extendible: Given a partial ladder diagram, with f_0 and g_0 strong embeddings and α_0, β_0, α_1 arbitrary embeddings,

\[
\begin{array}{c}
A_0 \\
\downarrow \alpha_0 \quad \downarrow \beta_0 \\
B_0 \\
\end{array} \quad f_0 \quad \alpha_1 \quad g_0 \quad \begin{array}{c}
A_1 \\
\downarrow \alpha_1 \\
B_1 \\
\end{array}
\]

already α_1 ∘ f_0 must be a strong embedding, since A_1 has more equivalence classes than A_0, and hence so does B_1.

K is not coherent, and hence not smooth: Let A be any structure in K with at least one class, let f: A → B be an embedding of A into a structure B in K which extends the equivalence classes of A but adds no new classes, and let g: B → C be an embedding of B into a structure C in K which adds elements in a new class. Then both g and g ∘ f are strong embeddings, but f is not.

**Remark 1.17.** One moral of Example 1.16 is that there are two easy ways to obtain extendibility: Smoothness says that strong substructure is closed downward (A ≤_K B and A ⊆ C ⊆ B implies A ≤_K C). But if strong substructure is closed upward instead (A ≺_K B and A ⊆ B ⊆ C implies A ≺_K C), this also implies that K is extendible.

**Example 1.18 (A class which is coherent but not extendible).** Let L be the empty language, and let K be the class of all finite sets. Say an embedding A → B is strong when |A| and |B| are both even or |A| and |B| are both odd.

K is coherent: If f: A → B and g: B → C are embeddings such that g is strong and g ∘ f are strong, then |B| and |C| have the same parity and |A| and |C| have the same parity, so |A| and |B| have the same parity. Hence f is strong.

K is not extendible, and hence not smooth: Build a ladder diagram such that each A_n has cardinality 2n and each B_n has cardinality 2n + 1. Then all the embeddings A_n → A_{n+1} and B_n → B_{n+1} are strong, but no embedding A_0 → B_n is strong.
Remark 1.19. Example 1.18 illustrates the somewhat strange behavior of non-extendible classes. In this example, a countable set which is constructed as a union of even-sized finite sets is different than a countable set which is constructed as a union of odd-sized finite sets: no bijection between them is strong. In an extendible class, on the other hand, the presentation of a structure as a direct limit carries no extra information that is not captured in the structure’s isomorphism type as an $L$-structure. In fact, for an extendible class $K$, every $K$-direct limit has a canonical maximal presentation, and hence any two presentations of the same structure as a $K$-direct limit fit together in a natural way. This is shown in Proposition 1.20, which is the key fact about extendible classes.

Proposition 1.20. Let $K$ be an extendible class, and let $M$ be a $K$-direct limit. The set of all finite strong substructures of $M$ together with the inclusions between them is a directed system $D_M$ in $\mathcal{C}_K$, the direct limit of which is $M$.

If $M$ and $N$ are $K$-direct limits, an embedding $f : M \to N$ is strong if and only if $f$ identifies $D_M$ with a subsystem of $D_N$.

Proof. To show that $D_M$ is a directed system, we just need to show that for any $A \preceq_K M$ and $B \preceq_K M$, there is some $C \preceq_K M$ with $A \preceq_K C$ and $B \preceq_K C$. Pick any presentation of $M$ as a $K$-direct limit, $M \cong \lim(C_i)$. Then there are some $i$ and $j$ such that $A \preceq_K C_i$ and $B \preceq_K C_j$. By directedness of the $(C_i)$ system, there is some $C_k$ with $A \preceq_K C_i \preceq_K C_k$ and $B \preceq_K C_j \preceq_K C_k$.

Clearly $M = \lim D_M = M$, since the maps in $D_M$ are inclusions between finite substructures of $M$, and for each $m \in M$, $m \in C_i$ for some $C_i \in D_M$.

For the second claim, consider an embedding $f : M \to N$ with $M \cong \lim D_M$ and $N \cong \lim D_N$. If $f : M \to N$ is strong, then for all $A$ in the system $D_M$, $f[A] \preceq_K N$, so $f[A]$ appears in the system $D_N$. $f$ clearly carries inclusions to inclusions, so $f$ identifies $D_M$ with a subsystem of $D_N$. Conversely, if for all $A$ in the system $D_M$, $f[A]$ is in the system $D_N$, then $f \upharpoonright A$ factors as the isomorphism $f_A : A \to f[A]$ followed by the inclusion of $f[A]$ into $N$. Both of these maps are strong, and hence $f$ is strong. \qed

1.3 Abstract elementary classes

We will now take a moment to observe that extendibility and coherence ensure that the class of $K$-direct limits, equipped with the strong substructure relation, is an abstract elementary class.

Definition 1.21. An abstract elementary class (AEC) in the language $L$ is a class $C$ of $L$-structures together with a relation $\preceq$ such that:

1. $\preceq$ is a partial order on $C$.
2. If $M \preceq N$, then $M$ is a substructure of $N$.
3. $C$ and $\preceq$ are closed under isomorphism: if $M \in C$ and $f : M \to M'$ is an isomorphism and $N \preceq M$, then $M' \in C$ and $\sigma(N) \preceq \sigma(M) = M'$.
4. If $M_1, M_2, M_3 \in C$, and $M_1 \subseteq M_2$, $M_2 \preceq M_3$, and $M_1 \preceq M_3$, then $M_1 \preceq M_2$. 

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(5) If \( \{M_\alpha \mid \alpha < \gamma\} \) satisfies \( M_\alpha \preceq M_\beta \) for all \( \alpha \leq \beta < \gamma \), then
   
   (a) \( \bigcup_{\alpha<\gamma} M_\alpha \in C \), and
   
   (b) if \( M_\alpha \preceq N \) for all \( \alpha < \gamma \), then \( \bigcup_{\alpha<\gamma} M_\alpha \preceq N \).

(6) There is an infinite cardinal \( \text{LS}(C) \), called the \textit{Löwenheim-Skolem number} of \( C \), such that if \( M \in C \) and \( A \subseteq M \), then there exists \( N \preceq M \) such that \( A \subseteq N \) and \( |N| \leq |A| + \text{LS}(C) \).

**Proposition 1.22.** Let \( K \) be an extendible and coherent strong embedding class. Then \((S_{K_\infty}, \preceq_K)\) is an AEC with \textit{Löwenheim-Skolem} number \( \aleph_0 \). In particular, this is true when \( K \) is smooth. If \( K \) is extendible but not coherent, only condition (4) fails.

**Proof.** The AEC axioms (1) and (2) hold simply because \( \mathcal{C}_{K_\infty} \) is a subcategory of \( \mathcal{C}_L \). (3) holds because the class of \( K \)-direct limits is closed under isomorphism, and hence so is the class of strong embeddings between them, as a consequence of extendibility.

(4): This is just coherence for \( K_\infty \). We have shown in Proposition 1.14 that coherence extends from \( K \) to \( K_\infty \).

(5): For (a), we must show that the direct limit of a chain of \( K \)-direct limits is again a \( K \)-direct limit. For each \( M_\alpha \), consider the maximal presentation of \( M_\alpha \) as a \( K \)-direct limit, as in Proposition 1.20, which is the directed system \( D_\alpha \) consisting of all finite strong substructures of \( M_\alpha \) and the inclusions between them. Then if \( \alpha < \beta \), each strong substructure of \( M_\alpha \) is also a strong substructure of \( M_\beta \), so \( D_\alpha \) is a subsystem of \( D_\beta \). Taking the union along the chain of directed systems \( \{D_\alpha \mid \alpha < \gamma\} \), we obtain a directed system \( D_\gamma \), the direct limit along which is \( \bigcup_{\alpha<\gamma} M_\alpha \).

For (b), observe that if \( D_N \) is the maximal presentation of \( N \) as a \( K \)-direct limit, then all of the directed systems \( D_\alpha \) are subsystems of \( D_N \). Then also \( D_\gamma \) is a subsystem of \( D_N \), so \( \bigcup_{\beta<\gamma} M_\alpha \preceq_K N \).

(6): Take \( \text{LS}(C) = \aleph_0 \). Let \( M \in K_\infty \) and \( A \subseteq M \). For each element \( a \in A \), pick some finite \( B_{\{a\}} \preceq_K M \) such that \( a \in B_{\{a\}} \). Now we close the family \( \{B_{\{a\}} \mid a \in A\} \) to a directed system of strong substructures of \( M \) by picking, for each nonempty finite \( X \subseteq A \), a finite structure \( B_X \preceq_K M \) such that \( B_Y \preceq_K B_X \) whenever \( Y \subseteq X \). This can be done by induction on the size of \( X \), using the fact that the strong substructures of \( M \) form a directed system. The direct limit \( N = \lim_{\alpha} \{B_X \mid X \subseteq \text{fin} \, A\} \) is a strong substructure of \( M \). If \( A \) is finite, then \( N \) is just \( B_A \), which is finite. If \( A \) is infinite, then there are \( |A| \) finite subsets of \( A \), each contributing finitely many elements to the direct limit, so \( |N| \leq |A| = |A| + \aleph_0 \).

As the proof of (6) shows, the Löwenheim-Skolem axiom holds in a strong form for the AEC \((S_{K_\infty}, \preceq_K)\). It is an example of what Baldwin, Koerwein, and Laskowski call a locally finite AEC.

**Definition 1.23.** An AEC \((C, \preceq)\) is \textit{locally finite} if for all \( M \in C \) and finite \( A \subseteq M \), there is a finite \( N \subseteq M \) with \( A \subseteq N \preceq M \). Equivalently, every structure in \( C \) is the direct limit of its finite strong substructures.

The term “finitary AEC” has also been used, by Hyttinen and Kesälä, for an altogether different notion. To avoid confusion, and as a demonstration of how tameness properties of AECs may interact with properties of the class \( K \), we give their definition here and explore which clauses must hold and which may fail in our context.
Definition 1.24 (Hyttinen and Kesälä). An AEC is finitary if

(1) It has Löwenheim-Skolem number $\aleph_0$,

(2) It has arbitrarily large models,

(3) It has the amalgamation property (if $f: A \to B$ and $g: B \to C$ are strong embeddings, then there exists $D$ and strong embeddings $f': B \to D$ and $g': C \to D$ such that $f' \circ f = g' \circ g$),

(4) It has the joint embedding property (given $A$ and $B$, there exists $C$ and strong embeddings $f: A \to C$ and $g: B \to C$), and

(5) It has finite character.

We have already noted that an AEC of the form $(S_{K_\infty}, \preceq_K)$ satisfies condition (1). I will now define finite character and show that condition (5) is also satisfied. But Examples 1.27 and 1.28 below will show that conditions (2), (3), and (4) may fail.

Definition 1.25. An AEC has finite character if it satisfies the following: If $M_1 \subseteq M_2$ and for all $\bar{a}$ from $M_1$, there exist $N$ and strong embeddings $f: M_1 \to N$ and $g: M_2 \to N$ such that $f(\bar{a}) = g(\bar{a})$, then $M_1 \preceq M_2$.

Proposition 1.26. If $K$ is an extendible and coherent strong embedding class, then the AEC $(S_{K_\infty}, \preceq_K)$ has finite character.

Proof. Let $M_1 \subseteq M_2$ satisfying the hypothesis on finite tuples from $M_1$. To show that $M_1 \preceq_K M_2$, we need to show that for all $A \preceq_K M_1$, $A \preceq_K M_2$. Let $\bar{a}$ enumerate $A$. Then we have strong embeddings $f: M_1 \to N$ and $g: M_2 \to N$ such that $f(\bar{a}) = g(\bar{a})$. Note that since $f$ is strong, $f[A] \preceq_K N$. Then we have $g[A] \subseteq g[M_2]$, $g[A] = f[A] \preceq_K N$, and $g[M_2] \preceq_K N$. By coherence, $g[A] \preceq_K g[M_2]$, and by isomorphism invariance, $A \preceq_K M_2$. \fbox{}

Example 1.27 (A smooth class $K$ such that $K_\infty$ has no uncountable models). Consider again the class $K'$ from Example 1.5 (finite connected acyclic graphs of degree at most two). $K'$ is clearly smooth, since all embeddings between structures in $K'$ are strong. Any structure in $K'_{\infty}$ is again a connected acyclic graph of degree at most two. But there are only two such infinite graphs, and both are countable: an infinite chain of vertices with one endpoint, $A_1$, and an infinite chain of vertices with no endpoints, $A_0$. The latter structure is maximal, in the sense that any strong embedding out of $A_0$ is an isomorphism.

Example 1.28 (A smooth class failing JEP and AP). Let $L = \{P\}$, where $P$ is a unary predicate, and let $K$ be the class of all finite sets such that every element is in $P$ or no element is in $P$. Call all embeddings between structures in $K$ strong. Note that if $f: A \to B$ is an embedding between nonempty structures in $K$, then $P$ is all of $A$ and all of $B$, or $P$ is empty in both $A$ and $B$. However, the empty structure (which is in $K$) embeds in every structure in $K$. $K$ is smooth since all embeddings are strong. But $K$ clearly fails to satisfy the joint embedding property and the amalgamation property (over the empty structure). Since $K$ is contained in $K_\infty$, $K_\infty$ also fails to satisfy these properties.
1.4 Definability concerns

(TO DO: Write this section - The point is that to get a handle on the model theory of the generic limit of \(K\), as in Section 3, one often needs definability conditions on \(\preceq_K\), and the definitions and discussion should go here. A name - “separability” - already exists in BKL for the notion that all isomorphisms types are determined by finitary q.f. formulas)

1.5 The space of countable labeled \(K\)-direct limits and genericity

Throughout this subsection, let \(K\) be a countable extendible strong embedding class. We will introduce a topological space of countable \(K\)-direct limits. The purpose of this space is to allow us to formulate the concept of a generic property of \(K\)-direct limits, in the sense of Baire category.

**Definition 1.29.** A labeled structure is one with domain \(\omega\) or an initial segment \([n] = \{0, \ldots, n - 1\} \subset \omega\). If \(M\) is a labeled structure with domain containing \([n]\), we denote by \(M[n]\) the induced substructure with domain \([n]\), and we call \(M[n]\) an initial segment of \(M\). We say that an infinite labeled \(K\)-direct limit \(M\) is well-labeled if for all \(n\) there exists \(m > n\) such that \(M[m] \preceq_K M\), i.e. \(M\) is a direct limit along a chain of initial segments of \(M\).

Let \(S_K\) be the set of all well-labeled \(K\)-direct limits. We topologize \(S_K\) by taking as generators the basic open sets \(O_A = \{M \in S_K \mid M[n] = A \preceq_K M\}\) for each \(n \in \omega\) and each finite labeled structure \(A\) in \(K\) with domain \([n]\). To better understand \(S_K\), we will show that it is homeomorphic to the space of paths through a tree.

**Definition 1.30.** If \(A \preceq_K B\) are finite labeled structures with domain \([n]\) and \([m]\) respectively, \(n < m\), we say that \(A \preceq_K B\) is a minimal labeled extension if there is no \(n < l < m\) such that \(A \preceq_K B[l] \preceq_K B\).

A (finite or infinite) chain \(A_0 \prec_K A_1 \prec_K \ldots\) is a well-labeled chain if \(A_i \prec_K A_{i+1}\) is a minimal labeled extension for each \(i\) and no proper initial segment of \(A_0\) is a strong substructure of \(A_0\).

Observe that each well-labeled \(K\)-direct limit \(M\) corresponds to a unique infinite well-labeled chain, consisting of all initial segments which are strong substructures of \(M\), and conversely the direct limit of any infinite well-labeled chain is a well-labeled \(K\)-direct limit.

Let \((\mathcal{T}_K, \leq)\) be the tree of all finite well-labeled chains, ordered by extension. An infinite path through the tree is simply an infinite well-labeled chain, and the bijection between well-labeled \(K\)-direct limits and infinite well-labeled chains gives a homeomorphism between \(S_K\) and the usual space of paths through \(\mathcal{T}_K\). Since \(K\) is countable, there are only countably many finite well-labeled chains, so the tree \(\mathcal{T}_K\) is countably branching, and the space \(S_K\) is a Polish space.

**Definition 1.31.** Let \(P\) be a property of well-labeled \(K\)-direct limits. We write \([P]\) for the set of all structures in \(S_K\) satisfying \(P\). We say \(P\) is generic if \([P]\) is comeager in \(S_K\).

We can analyze the genericity of \(P\) by defining the game \(G(P)\) for two players.
Definition 1.32. Given a property $P$ of well-labeled $K$-direct limits, we define a game for two players, $G(P)$. If $P$ is the property “isomorphic to $M$” for a countable $K$-direct limit $M$, we write $G(M)$ instead of $G(P)$.

Player I begins by selecting a labeled structure $A_i$ in $K$. The players then take turns selecting labeled structures $\{A_i \mid i \in \omega\}$ such that for all $i$, $A_i$ is in $K$, $A_i \prec_K A_{i+1}$, $A_i$ is a proper initial segment of $A_{i+1}$, and the open set $O_{A_i}$ is nonempty (it is illegal to choose a structure $A_i$ which cannot be strongly embedded in any infinite $K$-direct limit). The extension $A_i \prec_K A_{i+1}$ is not required to be minimal. After $\omega$ many turns, the players have constructed a chain $A_0 \prec_K A_1 \prec_K \ldots$. Player II wins if $M = \lim_{\omega}(A_i)$ is in $[P]$. Otherwise, Player I wins.

Definition 1.33. We say the property $P$ is determined if one of the players has a winning strategy in the game $G(P)$.

Fact 1.34. The game $G(P)$ is equivalent to the classical Banach-Mazur game on $S_K$ defined with payoff set $S_K \setminus [P]$ for Player I and the distinguished family of sets $\{O_A\}$. We recall the following facts (for a reference, see any book on descriptive set theory, e.g. Kechris):

1. If $[P]$ is Borel, or more generally if $[P]$ has the property of Baire, $P$ is determined.
2. Player II has a winning strategy in $G(P)$ if and only if $[P]$ is comeager in $S_K$, i.e. $P$ is generic.

Many reasonable properties of $K$-direct limits are determined.

Proposition 1.35. For every $A$ in $K$, let $\text{Emb}_A$ be the property that $A$ embeds strongly in $M$. Then the set $[\text{Emb}_A]$ is open. For every formula $\varphi(\overline{a})$ of $L_{\omega_1,\omega}$ and $\overline{a} \in \omega$, the set $[\varphi(\overline{a})] = \{M \in S_K \mid M \models \varphi(\overline{a})\}$ is Borel.

Proof. For the first claim, let $I = \{B \in K \mid B$ is labeled, and $A$ embeds strongly in $B\}$. Since any structure $M$ in $S_K$ is a direct limit along a chain of initial segments, any strong embedding of $A$ in $M$ factors through some initial segment $B = M_{[n]}$ appearing in this chain. Conversely, if $A$ embeds strongly in an initial segment which is a strong substructure of $M$, then it embeds strongly in $M$. Hence $[\text{Emb}_A] = \bigcup_{B \in I} [O_B]$.

We establish the second claim by induction on the complexity of $\varphi$.

If $\varphi$ is atomic, let $N = \max(\overline{a})$. Then $[\varphi(\overline{a})] = \bigcup_{A \models \varphi(\overline{a})} O_A$, where the union is taken over all finite labeled structures with domain $[n]$ such that $n > N$ and $A \models \varphi(\overline{a})$.

Countable disjunction and negation correspond to countable union and complement in $S_K$, and the Borel sets are closed under these operations.

If $\varphi$ is $\exists y \psi(\overline{a}, y)$, then $[\varphi(\overline{a})] = \bigcup_{b \in \omega} [\psi(\overline{a}, b)]$, a countable union. \hfill $\square$

Recall Scott’s Isomorphism theorem: For any countable structure $M$, there is a sentence $\varphi_M$ of $L_{\omega_1,\omega}$ such that if $N$ is a countable structure, then $N \models \varphi_M$ if and only if $N \cong M$. This means that the property “isomorphic to $M$” is Borel.

Question 1.36. The classical Lopez-Escobar theorem says that every Borel set in the space of countable labeled $L$-structures which is closed under isomorphism is the set of models for a sentence of $L_{\omega_1,\omega}$. Is there a Lopez-Escobar theorem in the space $S_K$?

(To Do: Is there anything relevant to say about how definability conditions on $K$ bring $S_K$ closer to being a nice subset of the usual space of countable labeled $L$-structures?)
2 Generalized Fraïssé theory

2.1 Amalgamation

Definition 2.1. Let $K$ be an extendible strong embedding class.

(1) $K$ has the joint embedding property if for all $A$ and $B$ in $K$ there exists $C$ in $K$ and strong embeddings $f: A \to C$ and $g: B \to C$:

$$
\begin{array}{c}
\text{C} \\
\downarrow f \\
A \\
\downarrow \uparrow \\
\text{B} \\
\downarrow g
\end{array}
$$

(2) A strong embedding $f: A \to B$ is an amalgamation embedding if for all strong embeddings $g_1: B \to C_1$ and $g_2: B \to C_2$, there is a structure $D$ in $K$ and strong embeddings $h_1: C_1 \to D$ and $h_2: C_2 \to D$ such that $h_1 \circ g_1 \circ f = h_2 \circ g_2 \circ f$:

$$
\begin{array}{c}
\text{D} \\
\downarrow h_1 \\
\text{C}_1 \\
\downarrow g_1 \\
\text{B} \\
\downarrow \uparrow \\
\text{C}_2 \\
\downarrow g_2 \\
\text{B} \\
\downarrow \uparrow \\
\text{A} \\
\downarrow f
\end{array}
$$

If the identity map $\text{id}_A$ is an amalgamation embedding, we say that $A$ is an amalgamation base.

(3) $K$ has the weak amalgamation property if for all $A$ in $K$ there exists a structure $B$ in $K$ and an amalgamation embedding $f: A \to B$.

(4) $K$ has the amalgamation property if every $A$ is an amalgamation base (i.e. we can always take $B = A$ and $f = \text{id}_A$ in the definition of the weak amalgamation property).

If $K$ has the joint embedding property and the weak amalgamation property, and there are only countably many structures in $K$ up to isomorphism, we say that $K$ is a generalized Fraïssé class.

Remark 2.2. The class of amalgamation embeddings is closed under isomorphism and stable under composition with strong embeddings. That is, if $g: B \to C$ is an amalgamation embedding, and $f: A \to B$ and $h: C \to D$ are strong embeddings, then $g \circ f$ and $h \circ g$ are amalgamation embeddings. Consequently, $K$ has the amalgamation property if and only if every strong embedding in $K$ is an amalgamation embedding: If $\text{id}_A$ is an amalgamation embedding, then for any $f: A \to B$, $f = f \circ \text{id}_A$ is an amalgamation embedding.
The easiest way to verify that a class has the weak amalgamation property is to find a cofinal subclass with the amalgamation property, as in the following proposition.

**Proposition 2.3.** If a strong embedding class $K$ contains a cofinal subclass $K'$ with the amalgamation property, then $K$ has the weak amalgamation property.

**Proof.** Let $A$ be in $K$, and let $f: A \rightarrow B$ be a $K$-embedding, with $B$ in $K'$. I claim that $f$ is an amalgamation embedding. Given $K$-embeddings $g_1: B \rightarrow C_1$ and $g_2: B \rightarrow C_2$, by cofinality we can find $K$-embeddings $g'_1: C_1 \rightarrow C'_1$ and $g'_2: C_2 \rightarrow C'_2$, with $C'_1$ and $C'_2$ in $K'$. Since $K'$ is a full subclass of $K$, $g'_1 \circ g_1$ and $g'_2 \circ g_2$ are strong embeddings in $K'$. Now $B$ is an amalgamation base in $K'$, so we can find $D$ in $K'$ and embeddings $h_1: C_1 \rightarrow D$ and $h_2: C_2 \rightarrow D$ such that $h_1 \circ (g'_1 \circ g_1) = h_2 \circ (g'_2 \circ g_2)$. But then $(h_1 \circ g'_1) \circ g_1 \circ f = (h_2 \circ g'_2) \circ g_2 \circ f$ shows that $D$ amalgamates $C_1$ and $C_2$ over $A$. □

In the proposition, we were able to find $B$ such that the following diagram commutes:

![Diagram]

This is a stronger condition (amalgamation over $B$) than what is required in the definition of amalgamation embedding (amalgamation over $A$). Intuitively, if $f: A \rightarrow B$ is an amalgamation embedding, then the way $A$ embeds into $B$ includes enough information about $A$ to ensure amalgamation over $A$. However, we may not yet have enough information about $B$ (i.e. $B$ may not be an amalgamation base). The following example shows that this situation is possible: there is a generalized Fraïssé class which does not contain a cofinal class with the amalgamation property. In fact, it may contain no amalgamation bases at all.

**Example 2.4.** Let $L = \{E, 1, 2, 3, 4, 5\}$, where $E$ is a binary relation and the other symbols are unary relations. Let $K$ be the class of finite nonempty connected acyclic graphs (with edge relation $E$), such that each vertex is colored by exactly one of the unary relations, and which omit the following five subgraphs:

```
1 ——— 2     2 ——— 3     3 ——— 4     4 ——— 5     5 ——— 1
  \   \         \   \         \   \         \   \         \   \    
  3     4     5     1     2
```

For the class of strong embeddings, we take all embeddings between structures in $K$.

We say a vertex labeled $i$ is determined if it has a neighbor labeled $i + 1$ or $i + 2$ (here addition is interpreted cyclically, so, for example, $4 + 2 = 1$). If a vertex in a structure $A$
is determined, say by having a neighbor labeled $i + 1$, it cannot be connected to a vertex labeled $i + 2$ in any extension of $A$.

If a structure $A$ contains a vertex $v$ labeled $i$ which is undetermined, then $A$ cannot be an amalgamation base. Indeed, we can embed $A$ into structures $B_1$ and $B_2$ by adding a neighbor of $v$ labeled $i + 1$ in $B_1$ and $i + 2$ in $B_2$, and these two embeddings cannot be amalgamated over $A$.

I claim that no structure in $K$ is an amalgamation base. Suppose for contradiction that $A$ is an amalgamation base. Then every vertex in $A$ is determined. Choose an arbitrary vertex $v_0$ ($A$ is nonempty). Then $v_0$ has a neighbor which determines it, call this neighbor $v_1$. Continue in this way, defining $\{v_i \mid i \in \omega\}$ such that $v_{i+1}$ determines $v_i$. Note that if $v_{i+1}$ determines $v_i$, then $v_i$ does not determine $v_{i+1}$ (this is why we needed five predicates). Since also $A$ contains no cycles, $\{v_i \mid i \in \omega\}$ is an infinite path through $A$, contradicting finiteness.

However, $K$ has the weak amalgamation property. Indeed, given $A$, let $B$ be a structure obtained from $A$ by adjoining a single new vertex determining $v$ for each undetermined vertex $v \in A$. Then any two extensions $C_1, C_2$ of $B$ can be amalgamated freely over $A$ (i.e. by not identifying any elements of $C_1 \setminus A$ and $C_2 \setminus A$, and adding no edge relations between these sets).

To finish the verification that $K$ is a generalized Fraïssé class, note that $K$ contains only countably many structures up to isomorphism (it is a class of finite structures in a finite relational language), and $K$ has the joint embedding property: we can embed $A$ and $B$ into the disjoint union $A \sqcup B \sqcup \{\ast\}$, ensuring connectedness by connecting the single new vertex $\ast$ arbitrarily to one vertex of $A$ and one vertex of $B$, and labeling $\ast$ by any legal $i$.

The weak amalgamation property was identified independently by Kechris and Rosendal and by Ivanov (who calls it the “almost amalgamation property”) as a necessary and sufficient condition for the existence of a generic (in the sense of Section 1.5) countable $K$-direct limit. This is Theorem 2.10 below. Of course, the weak amalgamation property corresponds to a weakening of the usual $K$-ultrahomogeneity property of Fraïssé limits, described in the next section.

2.2 Homogeneity

Definition 2.5. Let $M$ be a $K$-direct limit.

(1) The $K$-age of $M$ is the class of all structures in $K$ which embed strongly in $M$.

(2) $M$ is universal if every $A$ in $K$ embeds strongly in $M$ (i.e. the $K$-age of $M$ is $S_K$).

(3) $M$ is weakly-$K$-homogeneous if for all $A \preceq_K M$, there exists $B$ in $K$, such that $A \preceq_K B \preceq_K M$, and for all strong embeddings $g: B \to C$ such that $C$ is in the $K$-age of $M$, there exists a strong embedding $h: C \to M$ such that, naming the inclusions $i: A \to B$ and $j: B \to M$, we have $h \circ g \circ i = j \circ i$ (as in the diagram on the left). We say $i: A \to B$
witnesses weak-$K$-homogeneity for $A$.

$M$ is $K$-homogeneous if we can always take $B = A$ (as in the diagram on the right).

(4) $M$ is weakly-$K$-ultrahomogeneous if for all $A \preceq_K M$, there exists $B$ in $K$, such that $A \preceq_K B \preceq_K M$, and for all strong embeddings $g: B \to M$, there is an automorphism $\sigma \in \text{Aut}(M)$ such that, naming the inclusions $i: A \to B$ and $j: B \to M$, we have $\sigma \circ j \circ i = g \circ i$ (as in the diagram on the left). We say $i: A \to B$ witnesses weak-$K$-ultrahomogeneity for $A$.

$M$ is $K$-ultrahomogeneous if we can always take $B = A$ (as in the diagram on the right).

**Theorem 2.6.** If $M$ and $N$ are countable $K$-direct limits which are weakly-$K$-homogeneous and which have the same $K$-age, then for any $A \preceq_K B \preceq_K M$ such that $B$ witnesses weak-$K$-homogeneity for $A$, and any strong embedding $g: B \to N$, there is an isomorphism $\sigma: M \cong N$ such that, naming the inclusions $i: A \to B$ and $j: B \to M$, we have $\sigma \circ j \circ i = g \circ i$.

**Proof.** We go back and forth, building the isomorphism $\sigma$ as a union of partial isomorphisms $\{\sigma_k \mid k \in \omega\}$, such that $\sigma_k$ has domain $A_k \preceq_K M$ and range $A'_k \preceq_K N$. Along the way, we also define a sequence of partial isomorphisms $\{\tau_k \mid k \in \omega\}$, such that $\tau_k$ has domain $B_k \preceq_K M$ and range $B'_k \preceq_K M$, with $A_k \preceq_K B_k$ and $A'_k \preceq_K B'_k$, and $\tau_k \upharpoonright A_k = \sigma_k$. Further, we will ensure that if $k$ is even, $B_k$ witnesses weak-$K$-homogeneity for $A_k$ (in $M$), and if $k$ is odd, $B'_k$ witnesses weak-$K$-homogeneity for $A'_k$ (in $N$).

To begin, let $A_0 = A$, $B_0 = B$, $_0 = g[A]$, $B'_0 = g[B]$, $\sigma_0 = g \upharpoonright A$, and $\tau_0 = g$. By assumption, $B_0$ witnesses weak-$K$-homogeneity for $A_0$. And already $\sigma_0$ carries $A$ to $g[A]$, so this ensures that we will have $\sigma \circ j \circ i = g \circ i$ at the end of the day.

Enumerate $M = \{m_k \mid k \in \omega\}$ and $N = \{n_k \mid k \in \omega\}$. At stage $2k + 1$, we extend the given partial isomorphism, $\sigma_{2k}$, to a partial isomorphism $\sigma_{2k+1}$ which includes $n_k$ in its range $A'_{2k+1}$. We are given $\sigma_{2k}:A_{2k} \to A'_{2k}$ and $\tau_{2k}:B_{2k} \to B'_{2k}$. Choose $A'_{2k+1} \preceq_K N$ such that $B'_{2k} \preceq_K A_{2k+1}$ and $n_k \in A'_{2k+1}$, and choose $B'_{2k+1}$ witnessing weak-$K$-homogeneity
for \( A'_{2k+1} \) (so \( A'_{2k+1} \preceq_K B'_{2k+1} \preceq_K N \)). Let \( h : B_{2k} \to B'_{2k+1} \) be the composition of \( \tau_{2k} \) with the inclusion \( B'_{2k} \to B_{2k+1} \). Since \( 2k \) is even, \( B_{2k} \) witnesses weak-\( K \)-homogeneity for \( A_{2k} \), and \( B'_{2k+1} \) is in the age of \( M \), since \( M \) and \( N \) have the same \( K \)-age, so there exists a strong embedding \( l : B'_{2k+1} \to M \) over \( A_{2k} \). Let \( A_{2k+1} = l[A'_{2k+1}], \sigma_{2k+1} = (l \upharpoonright A'_{2k+1})^{-1}, B_{2k+1} = l[B'_{2k+1}] \), and \( \tau_{2k+1} = l^{-1} \).

The even stage \( 2k + 2 \) is similar, in the other direction. \( \square \)

**Corollary 2.7.** If \( M \) is a \( K \)-direct limit, \( M \) is weakly-\( K \)-homogeneous if and only if \( M \) is weakly-\( K \)-ultrahomogeneous. If these conditions hold, then given \( A \preceq_K B \preceq_K M \), \( B \) witnesses weak-\( K \)-homogeneity for \( A \) if and only if it witnesses weak-\( K \)-ultrahomogeneity for \( A \).

**Proof.** Suppose \( M \) is weakly-\( K \)-homogeneous. Then for any \( A \preceq_K M \), any \( B \) witnessing weak-\( K \)-homogeneity for \( A \) also witnesses weak-\( K \)-ultrahomogeneity for \( A \). Indeed, for any strong embedding \( g : B \to M \), taking \( N = M \) in theorem Theorem 2.6 (so \( N \) and \( M \) clearly have the same \( K \)-age), the isomorphism \( \sigma \) is the desired automorphism of \( M \).

Conversely, suppose \( M \) is weakly-\( K \)-ultrahomogeneous. Then for any \( A \preceq_K M \), any \( B \) witnessing weak-\( K \)-ultrahomogeneity for \( A \) also witnesses weak-\( K \)-homogeneity for \( A \). Indeed, for any strong embedding \( g : B \to C \), with \( C \) in the \( K \)-age of \( M \), there is some strong embedding \( l : C \to M \). Then \((l \upharpoonright g[B]) \circ g \) is a strong embedding \( B \to M \), so by weak-\( K \)-ultrahomogeneity, there is an automorphism \( \sigma \) of \( M \) such that \( \sigma^{-1} \circ ((l \upharpoonright g[B]) \circ g) \circ i = j \circ i \). Let \( h = (\sigma^{-1} \upharpoonright l[C]) \circ l \). This is a strong embedding \( C \to M \), and \( h \circ g \circ i = j \circ i \), as desired. \( \square \)

**Corollary 2.8.** If there exists a countable \( K \)-direct limit which is universal and weakly-\( K \)-homogeneous, it is unique up to isomorphism.

**Proof.** Suppose \( M \) and \( N \) are countable universal weakly-\( K \)-homogeneous \( K \)-direct limits. \( M \) and \( N \) have the same \( K \)-age, namely \( K' \). Let \( A \preceq_K M \) be any finite strong substructure, and let \( B \) witness weak-\( K \)-homogeneity for \( A \). Since \( N \) is universal, there is a strong embedding \( B \to N \), so by Theorem 2.6, \( M \) and \( N \) are isomorphic. \( \square \)

It is easy to check that in Theorem 2.6 and its corollaries, we may remove the word "weak" everywhere by taking all the witnesses of weak-\( K \)-(ultra)homogeneity to be identities.

### 2.3 Equivalences and the generic limit

**Theorem 2.9.** Let \( K \) be an extendible strong embedding class. The following are equivalent:

1. \( K \) is countable up to isomorphism, and \( K \) has the joint embedding property.
2. \( K \) has a cofinal chain class.
3. There is a countable \( K \)-direct limit which is universal.

**Proof.** (1) \( \to \) (2): Enumerate the isomorphism classes in \( K \) as \( \{A_i \mid i \in \omega\} \). We build a chain \( C_0 \preceq_K C_1 \preceq_K C_2 \preceq_K \ldots \) by induction. Let \( C_0 = A_0 \). Given \( C_n \), find some \( C_{n+1} \) in \( K \).
such that $C_n \preceq_K C_{n+1}$ and $A_{n+1} \preceq_K C_{n+1}$. Let $K'$ be the full subclass of $K$ with structures $S_{K'} = \{ A \in K \mid A \cong C_i \text{ for some } i \in \omega \}$.

Clearly $K'$ is a chain class, and $K'$ is cofinal in $K$: Given $B$ in $K$, $B$ is isomorphic to $A_i$ for some $i \in \omega$. Then $B$ embeds strongly in $C_i$, composing the isomorphism $B \cong A_i$ with the inclusion $A_i \to C_i$.

(2) → (3): Let $K'$ be the cofinal chain class, with chain $C_0 \preceq_K C_1 \preceq_K C_2 \preceq_K \ldots$. The inclusions in the chain are $K$-embeddings, since $K'$ is a subclass of $K$. Let $M$ be the direct limit of the chain. $M$ is a countable $K$-direct limit, and given $A \in K$, $A$ embeds strongly in some $C_i$, by cofinality of $K'$, so $M$ is universal.

(3) → (1): Let $M$ be the countable universal $K$-direct limit. Since every structure in $K$ is isomorphic to a strong substructure of $M$, and $M$ has only countably many finite subsets, $K$ is countable up to isomorphism. For the joint embedding property, let $A$ and $B$ be in $K$. We may identify $A$ and $B$ with strong substructures of $M$. By Proposition 1.20, we may find $C \preceq_K M$ such that $A \preceq_K C$ and $B \preceq_K B$, as desired. \hfill \Box

**Theorem 2.10.** Let $K$ be an extendible strong embedding class. The following are equivalent:

(1) $K$ is a generalized Fraïssé class.

(2) $K$ is countable up to isomorphism, and there is a countable $K$-direct limit $M$ such that the property “isomorphic to $M$” is generic (i.e. Player II has a winning strategy in the game $G(M)$ of Definition 1.32).

(3) There is a countable $K$-direct limit $M$ which is universal and weakly-$K$-homogeneous.

If these conditions hold, the structure $M$ in (2) and (3) is the same, and given $A \preceq_K B \preceq_K M$, $B$ witnesses weak-$K$-homogeneity for $A$ if and only if the inclusion $A \to B$ is an amalgamation embedding. We call the structure $M$ the *generic limit* of $K$.

**Proof.** (1) → (2) and (3): Let $K$ be a generalized Fraïssé class. We will describe a winning strategy for Player II in the game $G(P)$, where $P$ is the property “universal and weakly-$K$-homogeneous”. This will establish (2), since Player II can’t win $G(P)$ if there are no countable $K$-direct limits with property $P$. Further, since any two such $K$-direct limits are isomorphic, by Corollary 2.8, we will find that the strategy is also a winning strategy for Player II in the game $G(M)$, establishing (3).

Player II enumerates the isomorphism classes in $K$ as $\{ A_i \mid i \in \omega \}$, and also makes a list $\{ D_i \mid i \in \omega \}$ of diagrams of the form $D : A \to B \to C$, where $A \to B$ is an amalgamation embedding and $B \to C$ is a strong embedding, such that each of the countably many such diagrams appears, up to isomorphism, infinitely many times on the list.

On Player II’s turn $n$, the current play of the game is a labeled structure $S_n$ in $K$. First, Player II examines the diagram $D_n : A \to B \to C$ and makes a list $\{ f_i \}_{i=0}^{m-1}$ of the strong embeddings $f_i : B \to S_n$. Let $T'_0 = S_n$. Given $T'_i$ an extension of $S_n$, we have a strong embedding $f'_i : B \to T'_i$ (composing $f_i$ with the inclusion $S_n \to T'_i$) and $g : B \to C$ (coming from $D_n$). Since $A \to B$ is an amalgamation embedding, we can find $T'_{i+1}$ amalgamating $T'_i$ and $C$ over $A$. After handling each of the $f_i$ in turn, we let $T' = T'_m$. 

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Next, Player II picks a labeled structure $T''$ extending $T'$ such that $A_n$ embeds strongly in $T''$, by the joint embedding property. Finally, Player II chooses an amalgamation embedding $T'' \to T'''$ and plays a labeled copy of $T'''$ extending $S_n$. Call this structure $T_n$.

Consider the $K$-direct limit $M$ constructed by a play of the game in which Player II follows the above strategy. $M$ is the direct limit of a chain $S_1 \preceq_K T_1 \preceq_K S_2 \preceq_K T_2 \preceq_K \ldots$. First, I claim that $M$ is universal. Let $A$ be in $K$. Then $A$ is isomorphic to $A_n$ for some $n$. On Player II’s turn $n$, a copy of $A_n$ was embedded in the structure $T_n$, and hence in $M$.

Next, I claim that for every $A \preceq_K M$, there exists $A \preceq_K B \preceq_K M$ such that the inclusion $A \to B$ is an amalgamation embedding. Indeed, $A$ is a strong substructure of $S_n$ for some $n$. Then $A$ is a strong substructure of the $T''$ constructed on Player II’s turn $n$. The inclusion $T'' \to T''' = T_n$ is an amalgamation embedding, and amalgamation embeddings are stable under composition with strong embeddings by Remark 2.2, so the inclusion $A \to T_n$ is an amalgamation embedding.

Finally, I claim that if $A \preceq_K B \preceq_K M$ and the inclusion $A \to B$ is an amalgamation embedding, then it witnesses weak-$K$-homogeneity of $M$. Indeed, $B$ is a strong substructure of $S_n$ for some $n$. Suppose we have a strong embedding $B \to C$. Then the diagram $A \to B \to C$ appears up to isomorphism infinitely many times among the $D_n$, so there is some $m > n$ such that $D_m$ is isomorphic to $A \to B \to C$. Now $B$ is a strong substructure of $S_m$, so this inclusion was among the $f_i$ considered by Player II on turn $m$, and $T_n$, and hence $M$, contains a copy of $C$ embedded over $A$.

$(2) \to (1)$: First, assume for contradiction that $K$ does not have the joint embedding property. Then there are two structures $A$ and $B$ in $K$ which cannot be strongly embedded into any $C$ in $K$. Clearly we cannot strongly embed both $A$ and $B$ into $M$, otherwise we could find $C \preceq_K M$ such that $A \preceq_K C$ and $B \preceq_K C$ by Proposition 1.20. Say without loss of generality that $A$ does not embed strongly into $M$. Then Player I has a winning strategy in the game $G(M)$: simply start by playing a labeled copy of $A$ (recall that it is Player I’s goal to produce a $K$-direct limit which is not isomorphic to $M$). This is a contradiction.

Next, assume for contradiction that $K$ does not have the weak amalgamation property. Then there is a structure $A$ such that no strong embedding $A \to B$ is an amalgamation embedding. Let Players I and II play two copies of the game $G(M)$ at once, so on their turn, a player makes a move in both games. At the end of the joint game, two $K$-direct limits $M_1$ and $M_2$ have been built. By playing the winning strategy in each game, Player II can force both $M_1$ and $M_2$ to be isomorphic to $M$. The plan is to show for a contradiction that Player I has a strategy in the joint game which forces $M_1 \not\cong M_2$.

Player I begins by playing a labeled copy of $A$ in both games. We denote by $A^*$ the copy of $A$ which is an initial segment of the play in Game 1. On each subsequent turn for Player I, the play of the joint game has produced two labeled structures in $K$, $B_1$ and $B_2$. For each strong embedding $f: A^* \to B_2$ (of which there are finitely many), Player I adds $f$ to a queue.

Player I then pulls an embedding from the front of the queue. This embedding may have been added during this turn or a previous one; it is of the form $f': A^* \to B'_2$, where $B'_2$ is some initial segment of $B_2$. Composing $f'$ with the inclusion $B'_2 \to B_2$, we have a strong embedding $f_2: A^* \to B_2$. We also have the inclusion $f_1: A^* \to B_1$. Consider the following
amalgamation problem:

\[ \begin{array}{c}
B \\
\downarrow g_1 \quad \downarrow g_2 \\
B_1 \quad B_2 \\
\uparrow f_1 \quad \uparrow f_2 \\
A^* 
\end{array} \]

Case 1: There does not exist a structure \( B \) and strong embeddings \( g_1 \) and \( g_2 \) making the above diagram commute. Then Player I extends \( B_1 \) and \( B_2 \) arbitrarily.

Case 2: There exists \( B \) and strong embeddings \( g_1 \) and \( g_2 \) making the above diagram commute. Then we have a single embedding \( f: A^* \to B \) coming from the diagram. By assumption on \( A \), \( f \) is not an amalgamation embedding. Hence there exist strong embeddings \( h_1: B \to C_1 \) and \( h_2: B \to C_2 \) which cannot be amalgamated over \( A^* \). Player I extends \( B_1 \) and \( B_2 \) to labeled copies of \( C_1 \) and \( C_2 \).

Now consider a play of the joint game in which Player II follows the winning strategy for \( G(M) \) in both games, and Player I follows the strategy described above. Two structures \( B_1^* \) and \( B_2^* \) are produced, and as observed earlier, there is an isomorphism \( \varphi: M_1 \cong M_2 \), since both are isomorphic to \( M \). Then \( \varphi \) embeds the specified copy \( A^* \) of \( A \) strongly into \( M_2 \). Let \( f^* = \varphi \upharpoonright A^* \). Now \( f^*[A^*] \) is a strong substructure of some initial segment \( B^* \) which was the play of Game 2 on Player I’s turn. On this turn, Player I added \( f^*: A^* \to B^* \) to the queue. And at some later turn, \( f^* \) was pulled from the front of the queue. Let \((B_1^*, B_2^*)\) be the play of the joint game on that turn.

Now we have the inclusion \( f_1^*: A^* \to B_1^* \) and the map \( f_2^*: A^* \to B_2^* \) which is the composition of \( f^* \) with the inclusion \( B^* \to B_2^* \). We can solve the amalgamation problem by picking some \( B \preceq_K M \) such that \( \varphi[B_1^*] \preceq_K B \) and \( B_2^* \preceq_K B \), so we are in Case 2, and Player 2 advanced the play to \((C_1^*, C_2^*)\) which cannot be amalgamated over \( A^* \). But just as we were able to amalgamate \( B_1^* \) and \( B_2^* \) to \( B \) over \( A^* \), the isomorphism \( \varphi \) allows us to amalgamate \( C_1^* \) to \( C_2^* \) over \( A^* \). This is a contradiction.

(3) \( \to \) (1): By Theorem 2.9, the existence of a countable universal \( K \)-direct limit implies that \( K \) is countable up to isomorphism and has the joint embedding property. It remains to verify the weak amalgamation property.

Let \( A \) be in \( K \). Identifying \( A \) with a strong substructure of \( M \), we can find \( A \preceq_K B \preceq_K M \) such that \( B \) witnesses weak-\( K \)-homogeneity for \( A \). I claim that the inclusion \( f: A \to B \) is an amalgamation embedding. Indeed, let \( g_1: B \to C_1 \) and \( g_2: B \to C_2 \) be strong embeddings. By weak-\( K \)-homogeneity, we can embed \( C_1 \) and \( C_2 \) into \( M \) over \( A \), by \( h_1: C_1 \to M \) and \( h_2: C_2 \to M \). By Proposition 1.20, there is \( D \preceq_K M \) such that \( h_1[C_1] \preceq_K D \) and \( h_2[C_2] \preceq_K D \). Then \( D \) amalgamates \( C_1 \) and \( C_2 \) over \( A \).

Note that we have shown that every inclusion which witnesses weak-\( K \)-homogeneity is an amalgamation embedding. The converse was shown in the (1) \( \to \) (2) and (3) direction. \( \square \)

**Theorem 2.11.** If \( K \) is a generalized Fraïssé class and \( K' \) is a cofinal subclass of \( K \), then \( K' \) is also a generalized Fraïssé class, and the generic limits of \( K \) and \( K' \) are isomorphic.

**Proof.** Since \( K' \) is a subclass of \( K \), it is also countable up to isomorphism. For the joint embedding property, let \( A \) and \( B \) be in \( K' \). By the joint embedding property in \( K \), we can
find $C$ in $K$ and $K$-embeddings $A \rightarrow C$ and $B \rightarrow C$. Further, by cofinality, we can find $D$ in $K'$ and a $K$-embedding $C \rightarrow D$. Then $A$ and $B$ embed strongly in $D$ according to $K'$.

For the weak amalgamation property, for any $A$ in $K'$, pick $A \rightarrow B$ an amalgamation embedding in $K$. Then find $B \rightarrow B'$ with $B'$ in $K'$. Since amalgamation embeddings are stable under composition with strong embeddings, $A \rightarrow B'$ is an amalgamation embedding for $K$. It remains to show that this is also an amalgamation embedding in $K'$. Suppose $B \rightarrow C_1$ and $B \rightarrow C_2$ are $K'$-embeddings. Then we can find $D$ in $K$ amalgamating $C_1$ and $C_2$ over $A$. We’re done after further embedding $D$ into $D'$ in $K'$.

Now let $M'$ be the generic limit of $K'$. Then $M'$ is also a $K$-direct limit. It suffices to show that $M'$ is universal for $K$ and weakly-$K$-homogeneous, since then it is isomorphic to the generic limit of $K$ by Corollary 2.8. For any $A$ in $K$; $A$-embeds in some $B$ in $K'$ by cofinality, which $K'$-embeds in $M'$ by universality. So $A$ $K$-embeds in $M'$; and $M'$ is universal for $K$.

Let $A \preceq_K M'$. Since we can write $M'$ as a direct limit of a directed system of structures in $K'$, $A \preceq_K B \preceq_K M'$ for some $B$ in $K'$. Now find $B \preceq_{K'} C \preceq_{K'} M$ such that $C$ witnesses weak-$K'$-homogeneity for $B$. I claim that $C$ also witnesses weak-$K$-homogeneity for $A$. Indeed, suppose $C \rightarrow D$ is a $K$-embedding. Then we can further embed $D$ into $D'$ in $K'$, and by weak-$K'$-homogeneity, we can embed $D'$ into $M'$ over $B$. This also embeds $D$ into $M'$ over $A$.

**Example 2.12.** We revisit the classes in Example 1.5. Recall that $K$ is the class of finite acyclic graphs of degree at most two, together with all embeddings, and $K'$ is the full subclass of connected graphs in $K$.

$K$ does not have the amalgamation property: Consider the graph $A$ consisting of two disconnected vertices, $v$ and $w$. We can embed $A$ into graphs $B_2$ and $B_3$ in which $v$ and $w$ are connected by paths of length 2 and 3, respectively. But $B_2$ and $B_3$ cannot be amalgamated over $A$, since any way of doing so would introduce a cycle.

But $K'$ does have the amalgamation property: A structure $A$ in $K'$ is just a chain of some length $n$. Picking one side of $A$ to call the left, and one side to call the right, an embedding of $A$ into $B$ in $K'$ just extends the left side of the chain by $l$ vertices and the right side of the chain by $r$ vertices. Given two such embeddings $A \rightarrow B_1$ and $A \rightarrow B_2$, we can amalgamate them by ending $A$ on the left by the maximum of $l_1$ and $l_2$ and on the right by the maximum of $r_1$ and $r_2$.

Since $K'$ is cofinal in $K$, $K$ has the weak amalgamation property. Any embedding of $A$ in $K$ into a connected graph in $K'$ is an amalgamation embedding. $K$ and $K'$ are both generalized Fraïssé classes with the same generic limit: a chain, infinite in both directions.

## 2.4 Genericity in classes without the weak amalgamation property

**Proposition 2.13.** Let $K$ be an extendible strong embedding class which is countable up to isomorphism. Then $K$ has the joint embedding property if and only if for any isomorphism-invariant determined property $P$ (e.g. with the property of Baire), either $P$ or $\neg P$ is generic (i.e. Player II has a winning strategy in the game $G(P)$ or $G(\neg P)$ of Definition 1.32).

**Proof.** First, suppose $K$ has the joint embedding property. Let $P$ be an isomorphism-invariant determined property, and suppose that $P$ is not generic. Then Player I has a
winning strategy in the game $G(P)$. We will show that Player II has a winning strategy in the game $G(\neg P)$, showing that $\neg P$ is generic.

Suppose $A$ is Player I’s first move in the winning strategy for $G(P)$. In the play of $G(\neg P)$, on Player II’s first turn, Player I has played some structure $B$. By the joint embedding property, there is a structure $C$ in $K$ such that $A$ and $B$ embed strongly in $C$. Player II plays a labeled copy of $C$ extending $B$, and keeps in mind a relabling of $C$, $C^*$, which makes $A$ into an initial segment.

On all future turns, Player II takes the current play of the game, thinks of the initial segment $C$ as being relabeled to $C^*$, and plays according to the winning strategy for Player I in the game $G(P)$. In this way, Player II ensures that the labeled $K$-direct limit constructed by the play of the game satisfies $\neg P$, at least when the initial segment $C$ is relabeled to $C^*$. But the property $P$ is isomorphism invariant, so the relabeling of an initial segment is irrelevant. Hence Player II wins the game $G(\neg P)$.

Conversely, suppose that $K$ does not have the joint embedding property. Then there are two structures $A$ and $B$ in $K$ which do not jointly embed into any $C$ in $K$. Consider the property $\text{Emb}_A$ that $A$ embeds strongly in to $M$. $\text{Emb}_A$ is isomorphism invariant, and it was shown in Proposition 1.35 that $\text{Emb}_A$ is open, hence determined. However, Player I has a winning strategy in $G(\neg \text{Emb}_A)$ (begin by playing a labeled copy of $A$), and in $G(\text{Emb}_A)$ (begin by playing a labeled copy of $B$). Thus neither $\text{Emb}_A$ nor $\neg \text{Emb}_A$ is generic. \hfill $\square$

**Remark 2.14.** Since every sentence $\varphi$ of $L_{\omega_1, \omega}$ is an isomorphism-invariant Borel property, if $K$ has the joint embedding property, we can define a complete $L_{\omega_1, \omega}$ theory $T^K_{\text{gen}} = \{ \varphi \mid [\varphi] \text{ is generic} \}$. If $K$ does not have the weak amalgamation property, $T^K_{\text{gen}}$ does not have any countable models, since for any countable $M$, the property “isomorphic to $M$” is expressed by the Scott sentence of $M$ is $L_{\omega_1, \omega}$, and it is not generic. However, if we restrict to a countable fragment $F$ of $L_{\omega_1, \omega}$, such as first-order logic, $T^K_{\text{gen}} \cap F$ has countable models: any countable intersection of comeager sets in $S_K$ is comeager, in particular, nonempty.

Note that if $K$ has the joint embedding property but not the weak amalgamation property, no generic property (e.g. $[T^K_{\text{gen}} \cap F]$) can be satisfied by only countably many countable structures up to isomorphism. Indeed, for any countable set $\{ M_i \mid i \in \omega \}$ of $K$-direct limits, with Scott sentences $\varphi_{M_i}, \bigcup_{i \in \omega} [\varphi_{M_i}]$ is a countable union of meager sets, hence is meager, and not generic. However, we can ask a Vaught’s conjecture style question: Does every generic property have $2^{\mathfrak{c}}$ countable models up to isomorphism? The answer is yes.

**Proposition 2.15.** Suppose that $K$ does not have the weak amalgamation property. Then if $P$ is a generic property, $P$ is satisfied by $2^{\mathfrak{c}}$ many countable $K$-direct limits, up to isomorphism.

**Proof.** We begin with a single copy of the game $G(P)$. At the beginning of a turn for Player I, there may be many active games. Player I advances the play of all of them according to the “isomorphism-squashing” stragegy given in the (2)$\rightarrow$(1) direction of Theorem 2.10, ensuring in the long run that no pair of plays of any of the current games will result in isomorphic direct limits. At the end of the turn, we clone all current games, doubling the number of games. Player II simply plays in all active games according to a winning strategy for $G(P)$.

At the end of the day, we will have constructed a complete binary tree of games, and each of the $2^{\mathfrak{c}}$ many paths through this tree produces a $K$-direct limit. These are pairwise
nonisomorphic, due to the strategy of Player I, and they all satisfy \( P \), due to the strategy of Player II.

In Remark 2.14, it was noted that if \( K \) has the joint embedding property but not the weak amalgamation property, then \( T_{\text{gen}}^K \) has no countable models. In fact, assuming a definability condition, it has no models of any cardinality. Of course, Proposition 2.15 implies that for any countable fragment \( F \), \( T_{\text{gen}}^K \cap F \) has \( 2^{\aleph_0} \)-many countable models.

**Proposition 2.16.** Suppose \( K \) has the joint embedding property but not the weak amalgamation property, and assume that for every \( A \) in \( K \) there is a formula \( \varphi_A(\overline{x}) \) of \( L_{\omega_1,\omega} \) such that that property “\( M \models \varphi_A(\overline{a}) \) if and only if \( \overline{a} \) enumerates a copy of \( A \) and \( \overline{a} \preceq_K M \)” is generic. Then the complete \( L_{\omega_1,\omega} \) theory \( T_{\text{gen}}^K \) has no models of any cardinality.

**Proof.** Let \( F \) be the countable fragment of \( L_{\omega_1,\omega} \) generated by the formulas \( \varphi_A(\overline{x}) \) for all \( A \) in \( K \). Note that since there are countably many such \( \varphi_A \), and for each, it is generic that it has its intended meaning, restricting attention to countable \( K \)-direct limits such that all the \( \varphi_A \) have their intended meaning does not change the notion of genericity.

Now the failure of the weak amalgamation property is witnessed by some \( A \) in \( K \) such that for all embeddings \( A \to B \) in \( K \), there are embeddings \( B \to C_1 \) and \( B \to C_2 \) in \( K \) which cannot be amalgamated over \( A \). By the joint embedding property, the sentence \( \exists\overline{x} \varphi_A(\overline{x}) \) is generic. I claim that for every complete consistent \( F \)-type \( p(\overline{x}) \) containing \( \varphi_A(\overline{x}) \), the sentence \( \forall\overline{x} \neg \bigwedge_{\psi \in p} \psi(\overline{x}) \) is generic. This suffices to show that \( T_{\text{gen}}^K \) has no models of any cardinality, since any model contains some tuple \( \overline{a} \) witness \( \varphi_A(\overline{x}) \), and this tuple must realize one of the types \( p(\overline{x}) \), but this is forbidden by \( T_{\text{gen}}^K \).

For a given \( p \), we describe a strategy for Player II to ensure that no tuple realizes \( p \). On a given turn, Player II look at a strong embedding of \( A \) in the structure built so far, \( B \). By failure of weak amalgamation, there are embeddings of \( B \) in \( C_1 \) and \( C_2 \) which cannot be amalgamated over \( A \). Now the complete type \( p \) contains a list of all the formulas of the form \( \exists\overline{y} \varphi_C(\overline{x},\overline{y}) \) expressing that the copy of \( A \) enumerated by \( \overline{x} \) embeds strongly into a copy of \( C \), which embeds strongly into \( M \). At most one of \( C_1 \) and \( C_2 \) appear on this list, otherwise they could be consistently amalgamated over \( A \). If \( C_i \) does not appear on the list, Player II extends \( B \) to a copy of \( C_i \). In this way, Player II ensures that the copy of \( A \) under consideration does not realize \( p \), and in the infinite play of the game, Player II has time to handle every embedding of \( A \).

**Example 2.17.** Let \( L = \{E, P\} \), where \( E \) is a binary relation and \( P \) is a unary relation. Let \( K \) be the class of finite acyclic graphs (with edge relation \( E \)) of degree at most 2 (see Examples 1.5 and 2.12), such that the vertices are colored arbitrarily by \( P \).

\( K \) does not have the weak amalgamation property. Indeed, given a nonempty \( A \) in \( K \), and any embedding \( A \to B \) in \( K \), we can further embed \( B \) into \( C_1 \) by choosing a connected component of \( B \) containing a connected component of \( A \) and extending by one vertex labeled \( P \) on each side of this component of \( B \), and to \( C_2 \) by extending by one vertex labeled \( \neg P \) on each side of this component of \( B \). These extensions cannot be amalgamated over \( A \).

However, \( K \) is countable up to isomorphism and has the joint embedding property: Given \( A \) and \( B \) in \( K \), we can embed them both in \( C \) simply by connecting one end of \( A \) to one end of \( B \). As in Proposition 2.13, this is enough to provide a dichotomy between generic and
cogeneric properties. The generic first-order theory identified in Remark 2.14 is simply the complete theory of a union of chains, infinite in both directions, which embed every finite $P$-labeled chain. Further, while connectedness is not first-order expressible, it is a generic property.

**Remark 2.18.** It is worth noting a connection with forcing which is visible in Example 2.17. Forcing with the tree of labeled structures described in Section 1.5 gives a generic $K$-direct limit $M$ in a forcing extension $V[G]$ which has all generic properties present in $V$. If $K$ has the weak amalgamation property, then $M$ is isomorphic to a structure in $V$, the generic limit. If not, it isn’t isomorphic to any structure in $M$.

In Example 2.17, an infinite $K$-direct limit can be viewed as encoding an $Z$-indexed binary sequence (of $P$’s and $\neg P$’s), up to reversing the order of $Z$, i.e. a real, approximately. Forcing to add a generic $K$-direct limit is equivalent to forcing to add a Cohen real.

Similar lines of investigation has been pursued by Knight, Montalban, and Schweber in their paper *Computable structures in generic extensions*, and by others.

## 3 Model-theoretic properties of the generic limit

**Definition 3.1.** Let $K$ be a generalized Fraïssé class, and let $M$ be its generic limit. A function $l: \omega \to \omega$ is a weak Löwenheim-Skolem function for $K$ if for all finite subsets $A \subseteq M$, there exist $B \preceq_K C \preceq_K M$ such that $A \subseteq B$, the inclusion $B \to C$ is an amalgamation embedding, and $|C| \leq l(|A|)$.

As usual, we can give an equivalent condition which doesn’t mention the generic limit.

**Proposition 3.2.** The function $l$ is a weak Löwenheim-Skolem function for $K$ if and only if for all $D$ in $K$ and $A \subseteq D$, there exist $B, C, D' \in K$ such that $B \preceq_K C \preceq_K D'$, $D \preceq_K D'$, $A \subseteq B$, the inclusion $B \to C$ is an amalgamation embedding, and $|C| \leq l(|A|)$.

**Proof.** Suppose that $l$ is a weak Löwenheim-Skolem function for $K$. Let $D \in K$ and $A \subseteq D$. We embed $D$ strongly in the generic limit, $M$. Now by the definition of weak Löwenheim-Skolem function, there exist $B \preceq_K C \preceq_K M$ such that $A \subseteq B$, the inclusion $B \to C$ is an amalgamation embedding, and $|C| \leq l(|A|)$. By Proposition 1.20, we may pick $D' \preceq_K M$ such that $D \preceq_K D'$ and $C \preceq_K D'$, and we have the desired configuration.

Conversely, suppose the condition in the Proposition holds, and let $A$ be a finite subset of the generic limit $M$. Then there is some $E \preceq_K M$ such that $A \subseteq E$. Pick $E \preceq_K D \preceq_K M$ such that $D$ witnesses weak-$K$-homogeneity for $E$. Now we can find $B, C, D' \in K$ such that $B \preceq_K C \preceq_K D'$, $D \preceq_K D'$, $A \subseteq B$, the inclusion $B \to C$ is an amalgamation embedding, and $|C| \leq l(|A|)$. By weak-$K$-homogeneity, we can embed $D'$ in $M$ over $E$ by an embedding $f$. Then we still have $A \subseteq E \subseteq f[B]$, the inclusion $f[B] \to f[C]$ an amalgamation embedding, and $|f[C]| \leq l(|A|)$.

**Theorem 3.3.** Let $K$ be a generalized Fraïssé class with generic limit $M$. Then $\text{Th}(M)$ is $\aleph_0$-categorical if and only if

1. For all $n \in \omega$, $K_n = \{ A \in S_K \mid |A| = n \}$ is finite, and
(2) $K$ has a weak Löwenheim-Skolem function.

Proof. Suppose $\text{Th}(M)$ is $\aleph_0$-categorical. If (1) does not hold, then there is some $n$ such that $K_n$ is infinite. But then $M$ has infinitely many substructures of size $n$, so there are already infinitely many quantifier-free $n$-types realized in $M$, contradicting Ryll-Nardzewski.

For (2), we define $l(n)$ as follows. Let $S_n$ be the (finite) set of complete $n$-types realized in $M$. For each $p(\bar{x}) \in S_n$, choose a tuple $\bar{a}_p \in M$ realizing $p$, and let $k_p$ be minimal such that there exists $B \preceq_K C \preceq_K M$ with $\bar{a}_p \in B$, the inclusion $B \rightarrow C$ an amalgamation embedding, and $|C| = k_p$. Let $l(n) = \max_{p \in S_n} k_p$.

Now if $A \subseteq M$ with $|A| = n$, enumerating $A$ as $\bar{a}$, we have $M \models p(\bar{a})$ for some type $p(\bar{x}) \in S_n$. Since $\aleph_0$-categorical structures are strongly homogeneous, there is an automorphism $\sigma \in \text{Aut}(M)$ moving $\bar{a}$ to the realization $\bar{a}_p$ of $p$ chosen above. There exists $B \preceq_K C \preceq_K M$ with $|C| \leq l(n)$ such that $\bar{a}_p \in B$ and the inclusion $B \rightarrow C$ is an amalgamation embedding. Letting $C' = \sigma^{-1}(C)$, and $B' = \sigma^{-1}(B)$, we have $|C'| \leq l(n)$, $B' \preceq_K C' \preceq_K M$, the inclusion $B' \rightarrow C'$ an amalgamation embedding, and $A \subseteq B'$.

For the converse, we will show assuming (1) and (2) that only finitely many $n$-types are realized in $M$ for each $n$. For each $n$-tuple $\bar{a} \in M$, choose some $B_{\bar{a}} \preceq_K C_{\bar{a}} \preceq_K M$ such that $\bar{a} \in B_{\bar{a}}$, the inclusion $B_{\bar{a}} \rightarrow C_{\bar{a}}$ is an amalgamation embedding, and $|C_{\bar{a}}| \leq l(n)$. Given two such tuples $\bar{a}$ and $\bar{a}'$, if $f: C_{\bar{a}} \rightarrow C_{\bar{a}'}$ is an isomorphism with $f(\bar{a}) = \bar{a}'$, then there is an automorphism $\sigma \in \text{Aut}(M)$ extending $f \upharpoonright B_{\bar{a}}$. So $\sigma(\bar{a}) = \bar{a}'$, and $\text{tp}(\bar{a}) = \text{tp}(\bar{a}')$.

Hence the type of $\bar{a}$ is determined by the sizes of $B_{\bar{a}}$ and $C_{\bar{a}}$, the isomorphism types of $B_{\bar{a}}$ and $C_{\bar{a}}$ given their size, and the way $\bar{a}$ sits inside $B_{\bar{a}}$ given its isomorphism type. There are only finitely many choices for each of these data, so there are only finitely many $n$-types realized in $M$, so $\text{Th}(M)$ is $\aleph_0$-categorical by Ryll-Nardzewski. \hfill $\square$

(To do: Add stuff about trivial acl, quantifier-elimination, model-completeness, near-model-completeness, existential-closedness, atomicity of the generic limit, under definability assumptions from Section 1.4 on $\preceq_K$. It would also be nice to include some general information about Hrushovski constructions here, though that may deserve its own section later.)

4 Cone-robustness and super-robustness

As usual, let $K$ be an extendible strong embedding class. We now take up the question of when the theory of the generic limit of $K$ is pseudofinite. We introduce two sufficient conditions, cone-robustness and super-robustness. The latter condition was first isolated by Cameron Hill, and both definitions generalize the notion of a robust chain (of chain complexity 0) developed by Macpherson and Steinhorn. In fact, in the case that $K$ is a chain class, they are equivalent.

Definition 4.1. $K$ is cone-robust if for every formula $\varphi(\bar{x})$ there is some $A_\varphi \in K$ such that if $A_\varphi \preceq_K B \preceq_K C$ and $\bar{b} \in B$, $B \models \varphi(\bar{b})$ if and only if $C \models \varphi(\bar{b})$.

That is, in the cone above $A_\varphi$, strong embeddings are elementary with respect to $\varphi(\bar{x})$.

Definition 4.2. Given $A \in K$, the rank of $A$, $\text{rk}(A)$, is the largest $n$ such that there is a chain of strong proper substructures $A_0 \preceq_K A_1 \preceq_K \ldots \preceq_K A_n = A$. 

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Definition 4.3. $K$ is super-robust if for every formula $\varphi(\overline{x})$ there is a rank $e_\varphi$ such that if $\text{rk}(B) \geq e_\varphi$, $B \preceq_K C$, and $\overline{b} \in B$, then $B \models \varphi(\overline{b})$ if and only if $C \models \varphi(\overline{b})$.

That is, above rank $e_\varphi$, strong embeddings are elementary with respect to $\varphi(\overline{x})$.

Cone-robustness and super-robustness are closely related. We now show that the latter implies the former, they are equivalent in the case of chain classes, and they are equivalent up to cofinality.

Proposition 4.4. If $K$ is super-robust, then it is cone-robust.

Proof. Given a formulas $\varphi(\overline{x})$, let $e_\varphi$ be the rank provided by super-robustness. Let $A_\varphi$ be any structure with $\text{rk}(A_\varphi) \geq e_\varphi$ (in the degenerate case that $K$ does not contain structures of arbitrarily large rank, we can satisfy the definition trivially by choosing $A_\varphi$ of maximal rank). Then if $A_\varphi \preceq_K B \preceq_K C$ and $\overline{b} \in B$, we have $\text{rk}(C) \geq \text{rk}(B) \geq \text{rk}(A_\varphi) \geq e_\varphi$, so by super-robustness $B \models \varphi(\overline{b})$ if and only if $C \models \varphi(\overline{b})$. \qed

Proposition 4.5. If $K$ is a cone-robust chain class, then $K$ is super-robust.

Proof. Suppose $K$ is a chain class, witnessed by $A_0 \preceq_K A_1 \preceq_K A_2 \preceq_K \ldots$. If a substructure relation $A \preceq_K B$ is not proper, then $A \cong B$, so we may assume that all of the substructure relations appearing in the chain are proper. Then $K$ contains at most one structure of size $n$ up to isomorphism for each $n \in \omega$, and these sizes are strictly increasing along the chain.

Now I claim that $\text{rk}(A_n) = n$. Indeed, the chain $A_0 \prec_K A_1 \prec_K \ldots \prec_K A_n$ witnesses that $\text{rk}(A_n) \geq n$, and if $B_0 \prec_K B_1 \prec_K \ldots \prec_K B_m = A_n$ is a chain of proper strong substructures, the fact that the sizes of the $B_j$ are strictly increasing along the chain implies that $m \leq n$.

Given $\varphi(\overline{x})$, take $A_\varphi$ as in the definition of cone-robustness. If $A_\varphi \cong A_n$, taking $e_\varphi = n$ shows that $K$ is super-robust, since if $\text{rk}(B) \geq e_\varphi$, then $B \cong A_m$ for $m \geq n$, and $B$ is in the cone above $A_\varphi$. \qed

Proposition 4.6. If $K$ is cone-robust and $K'$ is a cofinal subclass of $K$, then $K'$ is cone-robust.

Proof. For any formula $\varphi$, just take $A'_\varphi$ in $K'$ to be any structure in $K'$ into which $A_\varphi$ in $K$ embeds strongly. Since the cone over $A'_\varphi$ is a subclass of the cone over $A_\varphi$, the structures $A'_\varphi$ witness that $K'$ is cone-robust. \qed

Theorem 4.7. Let $K$ have the joint embedding property. Then $K$ has a cofinal cone-robust subclass if and only if $K$ has a cofinal super-robust subclass.

Proof. Any cofinal super-robust subclass is cone-robust. Conversely, let $K'$ be a cofinal cone-robust subclass of $K$. Then (as in the proof of Theorem 2.11), $K'$ is countable and has the joint embedding property. By Theorem 2.9, $K'$ has a cofinal chain class $K''$, which is also cofinal in $K$. By Proposition 4.6, $K''$ is cone-robust. By Proposition 4.5, $K''$ is super-robust. \qed

Now that the relationship between cone-robustness and super-robustness has been sorted out, we will address some consequences. The inspiration for the following theorems comes entirely from Section 2.3 of Cameron Hill’s paper “On Super/robust Classes of Finite Structures”, though we find that we can get slightly better results with cleaner proofs under the weaker assumption of cone-robustness. Theorem 4.8 is the analog of Lemma 1.11 in that paper, and the proof is the same.
**Theorem 4.8.** Let $M$ be a universal $K$-direct limit. If $K$ is cone-robust, then for every formula $\varphi(\bar{x})$ there is some $A^*_\varphi \preceq_K M$ such that if $A^*_\varphi \preceq_K B \preceq_K M$ and $\bar{b} \in B$, $B \models \varphi(\bar{b})$ if and only if $M \models \varphi(\bar{b})$.

The converse holds if $K$ is a generalized Fraïssé class and $M$ is its generic limit.

**Proof.** Suppose $K$ is cone-robust. We will define $A^*_\varphi$ by induction on the complexity of $\varphi(\bar{x})$. Putting $\varphi(\bar{x})$ in prenex normal form, it suffices to handle quantifier free $\varphi(\bar{x})$ as a base case and existential quantification and negation as inductive steps.

If $\varphi(\bar{x})$ is quantifier free, take $A^*_\varphi$ to be any structure in $K$. Indeed, for any $B \preceq_K M$ and $\bar{b} \in B$, we have $B \models \varphi(\bar{b})$ if and only if $M \models \varphi(\bar{b})$.

Suppose $\varphi(\bar{x})$ is $-\psi(\bar{x})$. Let $A^*_\varphi$ witness cone-robustness of $K$ for $\varphi(\bar{x})$. We identify $A^*_\varphi$ with an isomorphic copy in $M$. We are also given $A^*_\psi \preceq_K M$ by induction. By Proposition 1.20, we can pick $A^*_\psi \preceq_K M$ such that $A^*_\psi \preceq_K A^*_\varphi$ and $A^*_\varphi \preceq_K A^*_\psi$. Consider $\bar{b} \in B$ such that $A^*_\varphi \preceq_K B \preceq_K M$.

If $M \models \varphi(\bar{b})$, then there is some $c \in M$ such that $M \models \psi(\bar{b}, c)$. By Proposition 1.7, there is some $C \preceq_K M$ such that $c \in C$ and $B \preceq_K C$. Since $A^*_\psi \preceq_K C$, we have $C \models \psi(\bar{b}, c)$, so $C \models \varphi(\bar{b})$. Then since $A^*_\varphi \preceq_K B \preceq_K C$, $B \models \varphi(\bar{b})$.

On the other hand, if $B \models \varphi(\bar{b})$, then there is some $c \in B$ such that $B \models \psi(\bar{b}, c)$. Since $A^*_\varphi \preceq_K B$, $M \models \psi(\bar{b}, c)$, so $M \models \varphi(\bar{b})$.

For the converse, suppose $K$ is a generalized Fraïssé class and $M$ is its generic limit, and that for each formula $\varphi(\bar{x})$ there is some $A^*_\varphi$ as above. Pick $A^*_\varphi$ such that $A^*_\varphi \preceq_K A^*_\varphi \preceq_K M$ and $A^*_\varphi$ witnesses weak-$K$-homogeneity for $A^*_\varphi$. Then $A^*_\varphi$ witnesses cone-robustness for $\varphi$. Indeed, suppose $A^*_\varphi \preceq_K B \preceq_K C$. Then we also have $A^*_\varphi \preceq_K B \preceq_K C$, and by weak-$K$-homogeneity, we can embed $C$ in $M$ over $A^*_\varphi$, so that $A^*_\varphi \preceq_K B' \preceq_K C' \preceq_K M$, with $B'$ and $C'$ isomorphic copies of $B$ and $C$. For any $\bar{b} \in B'$, we have $B' \models \varphi(\bar{b})$ if and only if $M \models \varphi(\bar{b})$ if and only if $C' \models \varphi(\bar{b})$, and hence for any $\bar{b} \in B$, $B \models \varphi(\bar{b})$ if and only if $C \models \varphi(\bar{b})$.

**Definition 4.9.** Let $M$ be a $K$-direct limit.

1. We define $K(M) = \{ A \preceq_{fin} M \mid A \preceq_K M \}$.

2. Given $A \preceq_K M$, we define $K_A(M) = \{ B \in K(M) \mid A \subseteq B \}$, the cone above $A$ in $M$.

3. We denote by Cone the filter on $K(M)$ generated by the cones. That is,

$$
\text{Cone} = \{ X \subseteq K(M) \mid K_A(M) \subseteq X \text{ for some } A \in K(M) \}.
$$

The fact that Cone is closed under intersection follows immediately from Proposition 1.20.

If $M$ is a $K$-direct limit, and $U$ is any ultrafilter on $K(M)$ extending Cone, then there is a canonical embedding $j: M \to \prod_{A \in K(M)} A/U$. Indeed, for each $A$, let $j_A: M \to A$ be any function which is the identity on all $a \in A$ and acts arbitrarily on the other elements of $M$. Let $j: M \to \prod_{A \in K(M)} A/U$ be the product of the maps $j_A$, composed with the quotient by the ultrafilter.
If $(j'_A)_{A \in K(M)}$ is any other family of maps chosen in this way, then the induced map $j'$ is equal to $j$. Indeed, for all $a \in M$, choosing any $A \preceq_K M$ with $a \in A$, $j(a)$ and $j'(a)$ agree on $K_A(M) \in \text{Cone}$. Similarly, to check that $j$ is an embedding, we just need to check that on a cone, it respects the relations in $L$.

Note that $j$ need not be an elementary embedding in general.

**Proposition 4.10.** If $M$ is a universal $K$-direct limit, and if the conclusion of Theorem 4.8 holds, then for any ultrafilter $U$ on $K(M)$ extending $\text{Cone}$, the canonical embedding $j : M \to \prod_{A \in K(M)} A/U$ is an elementary embedding.

**Proof.** Let $\bar{a} \in M$, and suppose $M \models \varphi(\bar{a})$. Let $A^*_\varphi \preceq_K M$ be as in Theorem 4.8. Pick $B \preceq_K M$ such that $\bar{a} \in B$ and $A^*_\varphi \preceq_K B$. Then for all $C \in K_B(M)$, we have $A^*_\varphi \preceq_K C$, so $C \models \varphi(\bar{a})$. Hence $\{C \in K(M) \mid C \models \varphi(j_C(\bar{a}))\} \supseteq K_B(M)$, so this set is in $U$, and $\prod_{A \in K(M)} A/U \models \varphi(j(\bar{a}))$ by Los’s Theorem.

Note that the converse of Proposition 4.10 need not hold, even if $M$ is the generic limit of $K$. The fact that $j$ is an elementary embedding for any $U$ extending $\text{Cone}$ implies that for any formula $\varphi(\bar{b})$ holding in $M$, $\varphi(\bar{b})$ holds on the cone above some $A \preceq_K M$, but the cone in question may depend on the elements $\bar{b}$, not merely on the formula $\varphi$.

However, the following theorem allows us to complete the loop of implications under the hypotheses that $M$ is the generic limit of $K$ and $\text{Th}(M)$ is model-complete. Often, we’ll have the stronger condition that $\text{Th}(M)$ eliminates quantifiers, but the observation here is that model-completeness is sufficient.

**Theorem 4.11.** Let $K$ be a generalized Fraïssé class with generic limit $M$ such that $\text{Th}(M)$ is model-complete. Suppose further that for every sentence $\varphi \in \text{Th}(M)$ there is some $A^*_\varphi \preceq_K M$ such that if $A^*_\varphi \preceq_K B \preceq_K M$, $B \models \varphi$. Then $K$ is cone-robust.

**Proof.** Let $\varphi(\bar{x})$ be a formula. Since $\text{Th}(M)$ is model-complete, there is a universal formula $\psi$ and an existential formula $\psi'$ such that $\theta_{\varphi} : \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}) \leftrightarrow \psi'(\bar{x}))$ is in $\text{Th}(M)$.

To witness cone-robustness, pick $A^*_\varphi$ such that $A^*_\varphi \preceq_K A^*_\varphi \preceq_K M$ and $A^*_\varphi$ witnesses weak-$K$-homogeneity for $A^*_\theta_{\varphi}$. Indeed, suppose $A^*_\varphi \preceq_K B \preceq_K C$. By weak-$K$-homogeneity, there is an isomorphic copy $C'$ of $C$ strongly embedded in $M$ over $A^*_\theta_{\varphi}$, so that $A^*_\theta_{\varphi} \preceq_K B' \preceq_K C' \preceq_K M$.

Now $B' \models \theta_{\varphi}$, so for any $\bar{b} \in B'$, if $B' \models \varphi(\bar{b})$, then $B' \models \psi'(\bar{b})$, so $C' \models \psi'(\bar{b})$, as $\psi'$ is existential. And $C' \models \theta_{\varphi}$, so $C' \models \varphi(\bar{b})$. Similarly, if $C' \models \varphi(\bar{b})$, then $C' \models \psi(\bar{b})$, so $B' \models \psi(\bar{b})$, as $\psi$ is universal, so $B' \models \varphi(\bar{b})$. Of course, the same is true for $B$ and $C$.

In summary, we have

**Theorem 4.12.** If $K$ is a generalized Fraïssé class with generic limit $M$, and if $\text{Th}(M)$ is model-complete, then the following are equivalent:

1. $K$ is cone-robust.
2. For every formula $\varphi(\bar{x})$ there is some $A^*_\varphi \in K(M)$ such that if $A^*_\varphi \preceq_K B \preceq_K M$ and $\bar{b} \in B$, $B \models \varphi(\bar{b})$ if and only if $M \models \varphi(\bar{b})$.  

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(3) For any ultrafilter $U$ on $K(M)$ extending $\text{Cone}$, the canonical embedding $j: M \to \prod_{A \in K(M)} A/U$ is an elementary embedding.

(4) For every sentence $\varphi \in \text{Th}(M)$, there is some $A^* \in K(M)$ such that for all $A^*_\varphi \preceq_K B \preceq_K M$, $B \models \varphi$. Equivalently, $M \equiv \prod_{A \in K(M)} A/U$ for any ultrafilter $U$ on $K(M)$ extending $\text{Cone}$.

Proof. (1) $\to$ (2) is Theorem 4.8.

(2) $\to$ (3) is Proposition 4.10.

(3) $\to$ (4): Since condition (3) obviously implies $M \equiv \prod_{A \in K(M)} A/U$ for any ultrafilter $U$ on $K(M)$ extending $\text{Cone}$, we just need to verify the equivalence asserted in condition (4).

One direction is clear by Los’s Theorem, since the cone above $A^*_\varphi$ is in any such ultrafilter $U$. Conversely, if there is some $\varphi \in \text{Th}(M)$ which is not true on any cone, then $\{A \in K(M) \mid A \models \varphi\}$ is not in the $\text{Cone}$ filter, and hence there is some ultrafilter $U$ extending $\text{Cone}$ such that $\{A \in K(M) \mid A \models \neg \varphi\} \in U$. But then $M \not\equiv \prod_{A \in K(M)} A/U$.

(4) $\to$ (1) is Theorem 4.11.

In the next theorem, we just apply Theorem 4.12 to a cofinal cone-robust subclass. The only exceptions are the addition of conditions (0) (by Theorem 4.7) and a new condition (5).

**Theorem 4.13.** If $K$ is a generalized Fraïssé class with generic limit $M$, and if $\text{Th}(M)$ is model-complete, then the following are equivalent:

(0) $K$ has a cofinal super-robust subclass.

(1) $K$ has a cofinal cone-robust subclass.

(2) $K$ has a cofinal subclass $K'$ such that for every formula $\varphi(\bar{v})$ there is some $A^*_\varphi \in K'(M)$ such that if $B \in K'$ and $A^*_\varphi \preceq_K B \preceq_K M$ and $\bar{b} \in B$, $B \models \varphi(\bar{b})$ if and only if $M \models \varphi(\bar{b})$.

(3) $K$ has a cofinal subclass $K'$ such that for any ultrafilter $U$ on $K'(M)$ extending $\text{Cone}$, the canonical embedding $j: M \to \prod_{A \in K'(M)} A/U$ is an elementary embedding (i.e. $M$ is “age pseudo-finite with the embedding property”).

(4) $K$ has a cofinal subclass $K'$ such that for any ultrafilter $U$ on $K'(M)$ extending $\text{Cone}$, $M \equiv \prod_{A \in K'(M)} A/U$ (i.e. $M$ is “age pseudo-finite”).

(5) For all $\varphi \in \text{Th}(M)$, the models of $\varphi$ are cofinal in $M$ (i.e. $M$ has the “cofinal finite submodel property”). That is, for any finite subset $A \subseteq M$, there is some $B \preceq_K M$ such that $A \subseteq B$ and $B \models \varphi$.

Proof. (0) $\leftrightarrow$ (1) is Theorem 4.7.

(1) $\to$ (2) $\to$ (3) $\to$ (4) is by Theorem 4.12.

(4) $\to$ (5): Suppose $K$ has a cofinal subclass $K'$ as in (4). Then for any $\varphi \in \text{Th}(M)$, $\varphi$ is true on a cone in $K'(M)$, say on $K'_A(M)$. Since $K'$ is cofinal in $K$, we can write $M$ as a $K'$-direct limit, so for any finite subset $X \subseteq M$, there is some $B \preceq_K M$ such that $A \preceq_K B$ and $X \subseteq B$, and $B \models \varphi$, as desired.
(5) → (1): We build a cofinal chain class in $K$ which is cone-robust. Enumerate $\text{Th}(M)$ as $\{\varphi_i \mid i \in \omega\}$, and enumerate $K(M)$ as $\{B_i \mid i \in \omega\}$. Begin by picking any $A_0$ in $K(M)$ such that $A_0 \models \varphi_0$ and $B_0 \preceq_K A_0$.

At stage $n$, let $\psi_n = \bigwedge_{j=0}^n \varphi_n \in \text{Th}(M)$. Choose any $A_n$ in $K(M)$ such that $A_{n-1} \preceq_K A_n$, $B_n \preceq_K A_n$, and $A_n \models \psi_n$. This can be done, since the models of $\psi_n$ are cofinal in $M$.

The resulting chain class $K'$ is a generalized Fraïssé class with the same generic limit $M$ by Theorem 2.11. $\text{Th}(M)$ is model-complete, and every sentence $\psi_n \in \text{Th}(M)$ is true on a cone (above $A_n$) in $K'$, so by Theorem 4.11, $K'$ is cone-robust. \qed