MATH 32 FALL 2012
FINAL EXAM - PRACTICE EXAM SOLUTIONS

(1) You cut a slice from a circular pizza (centered at the origin) with radius 6" along radii at angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with the positive horizontal axis.

(a) (3 points) What is the area of your slice?

Solution: The interior angle of the slice is

$$\frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{12}.$$

So the area of the slice is

$$\frac{1}{2}\theta r^2 = \frac{1}{2}\frac{\pi}{12}6^2 = \frac{36\pi}{24} = \frac{3\pi}{2}.$$

(b) (3 points) What is the arc length of the outer portion of crust on your slice?

Solution:

$$r\theta = 6\frac{\pi}{12} = \frac{\pi}{2}.$$

(2) (6 points) Find all values of θ in the interval $[0, 2\pi]$ satisfying

$$\sin^2(\theta) + \frac{1}{2}\cos(\theta) = 1.$$

Solution: Rewrite $\sin^2(\theta) = 1 - \cos^2(\theta)$. Then, noticing that we have a quadratic in $\cos(\theta)$, write $x = \cos(\theta)$.

$$\begin{aligned}
(1 - \cos^2(\theta)) + \frac{1}{2}\cos(\theta) &= 1 \\
-\cos^2(\theta) + \frac{1}{2}\cos(\theta) &= 0 \\
-x^2 + \frac{1}{2}x &= 0 \\
-x(x - \frac{1}{2}) &= 0
\end{aligned}$$

So $x = 0$ or $\frac{1}{2}$, and so $\cos(\theta) = 0$ or $\frac{1}{2}$. For each of these two cosine values, there are two corresponding points on the unit circle. Since we only want angle measures between 0 and 2π, each of these points corresponds to only one angle, so there should be four solutions. They are

$$\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}.$$

(3) (3 points) Find an equation for the line perpendicular to $y = \frac{1}{3}x + 7$ through the point $(8, 26)$.

Solution: This line should have slope -3. In point-slope form,

$$y - 26 = -3(x - 8).$$

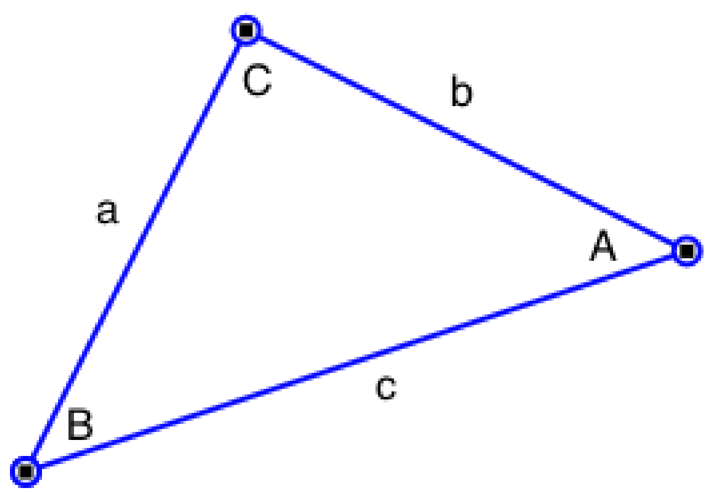
This is an acceptable solution. You may also rewrite the answer as

$$y = -3x + 50.$$

(4) (3 points) What is $\frac{\pi}{10}$ radians in degrees?

Solution: $\frac{\pi}{10} \cdot \frac{180°}{\pi} = \frac{180°}{10} = 18°$.

(5) In the triangle below, let $A = \frac{\pi}{6}$, $B = \frac{\pi}{8}$, and $a = 5$.



(a) (3 points) Find $\sin(B)$.

Solution: This is just asking us to find $\sin \frac{\pi}{8}$, which we can do using the half-angle formula.

$$\sin \frac{\pi}{8} = \sqrt{\frac{1 - \cos \frac{\pi}{4}}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}.$$

(b) (3 points) Find b.

Solution: By the law of sines,

$$\begin{aligned}\frac{\sin A}{a} &= \frac{\sin B}{b}\\ \frac{\sin\frac{\pi}{6}}{5} &= \frac{\sin\frac{\pi}{8}}{b}\\ \frac{\frac{1}{2}}{5} &= \frac{\sin\frac{\pi}{8}}{b}\\ \frac{1}{10}b &= \sin\frac{\pi}{8}\\ b &= 10\cdot\frac{\sqrt{2-\sqrt{2}}}{2}\\ b &= 5\sqrt{2-\sqrt{2}}.\end{aligned}$$

(6) Consider the rational function

$$f(x) = \frac{x^2-7x+12}{3x^2}$$

(a) (3 points) Does f have a horizontal asymptote? If so, what is it?

Solution: Yes, $y = \frac{1}{3}$.

(b) (6 points) Solve the inequality $f(x) \leq 0$.

Solution: Factoring the numerator, we have

$$f(x) = \frac{(x-3)(x-4)}{3x^2}.$$

f could switch sign at $x = 0$, $x = 3$, or $x = 4$. We'll do some sign analysis.

	$(-\infty, 0)$	$(0, 3)$	$(3, 4)$	$(4, \infty)$
$(x-3)$	$-$	$-$	$+$	$+$
$(x-4)$	$-$	$-$	$-$	$+$
$3x^2$	$+$	$+$	$+$	$+$
$f(x)$	$+$	$+$	$-$	$+$

So $f(x) < 0$ when $3 < x < 4$, and $f(x) = 0$ when $x = 3$ or $x = 4$, so the solutions are $3 \leq x \leq 4$.

(7) (6 points) Simplify the following expression:

$$e^{\frac{1}{2}\ln(x+3)-2\ln(x+1)}.$$

Solution:

$$\begin{aligned}e^{\frac{1}{2}\ln(x+3)-2\ln(x+1)} &= e^{\ln((x+3)^{\frac{1}{2}})}e^{\ln((x+1)^{-2})}\\ &= \sqrt{x+3}\cdot\frac{1}{(x+1)^2}\\ &= \frac{\sqrt{x+3}}{(x+1)^2}\end{aligned}$$

(8) (6 points) Show that for all θ, $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$.

Solution: We'll apply the angle sum formula and the double angle formulas.

$$\begin{aligned}\sin(3\theta) &= \sin(2\theta + \theta)\\ &= \sin(2\theta)\cos(\theta) + \cos(2\theta)\sin(\theta)\\ &= (2\sin(\theta)\cos(\theta))\cos(\theta) + (1 - 2\sin^2(\theta))\sin(\theta)\\ &= 2\sin(\theta)\cos^2(\theta) + \sin(\theta) - 2\sin^3(\theta)\\ &= 2\sin(\theta)(1 - \sin^2(\theta)) + \sin(\theta) - 2\sin^3(\theta)\\ &= 2\sin(\theta) + \sin(\theta) - 2\sin^3(\theta) - 2\sin^3(\theta)\\ &= 3\sin(\theta) - 4\sin^3(\theta)\end{aligned}$$

Note that since the end goal was an expression just involving sin, we chose the cosine double angle formula involving sine: $\cos(2\theta) = 1 - 2\sin^2(\theta)$. We also used the Pythagorean Identity in the form $\cos^2(\theta) = 1 - \sin^2(\theta)$ to transform the remaining cosine terms to sines.

(9) (a) (3 points) Find an equation for a circle with center $(2, -3)$ and radius 5.

Solution: $(x - 2)^2 + (y + 3)^2 = 25$.

(b) (3 points) What is the circumference of this circle?

Solution: $2\pi r = 2\pi \cdot 5 = 10\pi$.

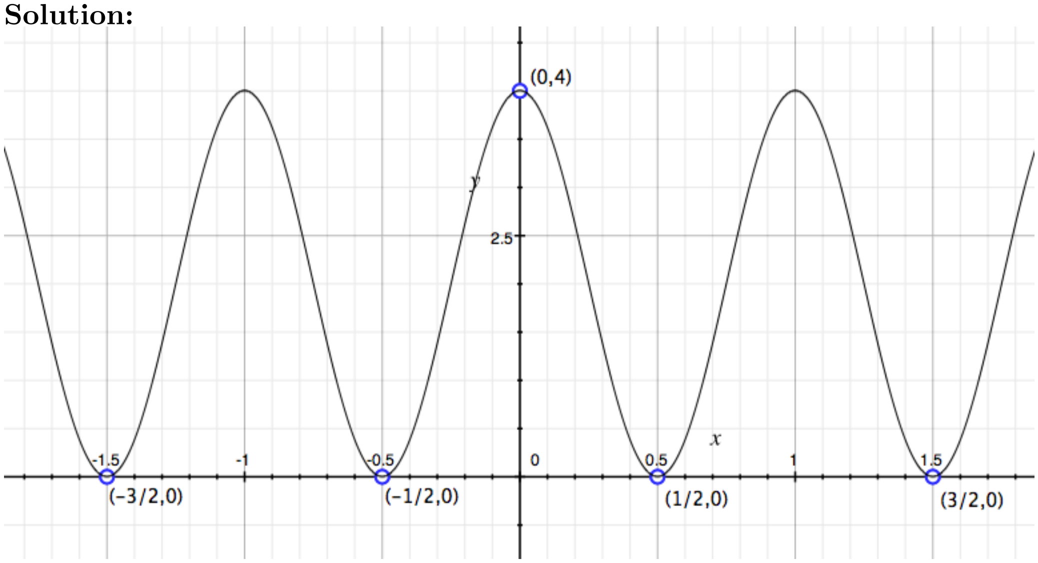
(c) (3 points) What is its area?

Solution: $\pi r^2 = \pi \cdot 5^2 = 25\pi$.

(10) Consider the function $f(x) = 2\cos(2\pi x) + 2$.

(a) (6 points) Sketch a graph of this function. Clearly label the y-intercept and several x-intercepts.

Solution:



(b) (3 points) What is the amplitude of this function?

Solution: The amplitude is 2.

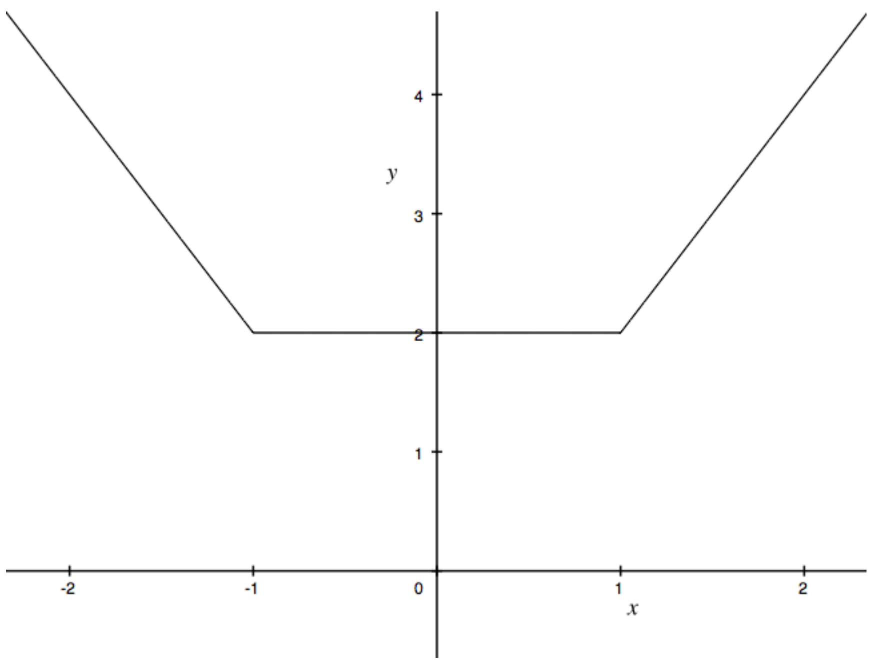
(c) (3 points) What is the period of this function?

Solution: The period is $\frac{2\pi}{2\pi} = 1$.

(11) (6 points) Sketch a graph of $y = |x-1| + |x+1|$. *Hint:* Write this as a piecewise function with three cases.

Solution:

$$y = \begin{cases} -(x-1) + -(x+1), \text{ if } x < -1 \\ -(x-1) + (x+1) \text{ if } -1 \le x < 1 \\ (x-1) + (x+1) \text{ if } 1 < x \end{cases} = \begin{cases} -2x, \text{ if } x < -1 \\ 2 \text{ if } -1 \le x < 1 \\ 2x \text{ if } 1 < x \end{cases}$$



(12) You put \$50 in a bank account with 8% interest compounded 4 times per year.

(a) (3 points) Write down an expression for the amount of money you will have after t years.

Solution:

$$A = 50\left(1 + \frac{.08}{4}\right)^{4t}$$

(b) (3 points) After how many years will you have \$80?

Solution: We'll solve the following for t:

$$80 = 50\left(1 + \frac{.08}{4}\right)^{4t}$$
$$\frac{80}{50} = (1.02)^{4t}$$
$$\log_{1.02}\left(\frac{8}{5}\right) = 4t$$
$$t = \frac{1}{4}\log_{1.02}(1.6)$$

(13) Evaluate the following:

(a) (3 points) $\cos(\cos^{-1}(.8))$

Solution: .8

(b) (3 points) $\sin^{-1}(\sin(\frac{13\pi}{16}))$

Solution: $\frac{13\pi}{16}$ is in the second quadrant, so it is not a possible output of $\sin^{-1}$. The angle in the first quadrant with the same sine value is $\pi - \frac{13\pi}{16} = \frac{3\pi}{16}$.

(c) (3 points) $\cos(\tan^{-1}(\frac{7}{5}))$

Solution: Draw a right triangle and label one angle $\theta = \tan^{-1}(\frac{7}{5})$. Label the opposite side 7 and the adjacent side 5. Then the hypoteneuse c satisfies $c^2 = 5^2 + 7^2$, so $c = \sqrt{74}$.
Then $\cos(\tan^{-1}(\frac{7}{5})) = \cos(\theta) = \frac{5}{\sqrt{74}}$.

(14) (3 points) Find $\log_{16}(32)$.

Solution: Change of base formula: $\log_{16}(32) = \frac{\log_2(32)}{\log_2(16)} = \frac{5}{4}$.