Example of a simple epsilon-delta proof

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Prove: $f(x) = \frac{1}{x}$ is continuous at $x = 4$.
Proof: Since $f(4)$ is defined and equals $\frac{1}{4}$, we just have to show that

$$\lim_{x \to 4} f(x) = f(4) = \frac{1}{4}.$$

We recall what this means, namely we're plugging in $a = 4$, $L = f(4) = \frac{1}{4}$, $f(x) = \frac{1}{x}$ into the definition of having a limit:
For each $\epsilon > 0$, we must find $\delta > 0$ so that whenever

$$0 \neq |x - 4| < \delta,$$

then we also have

$$|\frac{1}{x} - \frac{1}{4}| < \epsilon.$$

The δ-condition means that x is within distance δ of 4, and the ϵ-condition mandates that the value of the function be within distance ϵ of $f(x)$.
We hence see that this is a precise mathematical formulation of

> "getting $f(x)$ close to L by making x close to a".

Remark: Note also that the condition $0 \neq |x - 4|$ can be dropped because we picked as our L the value $f(4)$, so the additional constraint that comes from allowing $x = 4$ is the equation $|f(4) - f(4)| < \epsilon$, but of course the left hand side is zero. If this confuses you, just ignore it: It will not play a role. However, I will ignore the condition for the rest of the proof.

Now let's try to find a δ (depending on ϵ) that works. For this, we'll try to write down the left hand side of the ϵ-condition and rewrite it so that the left hand side of the δ-condition shows up:

$$|\frac{1}{x} - \frac{1}{4}| = |\frac{4 - x}{4x}| = \frac{|4 - x|}{|4x|}.$$

Now we know that the ϵ-condition needs only to be checked for $|x - 4| = |4 - x| < \delta$, so we can use this information to estimate further

$$\frac{|4 - x|}{|4x|} \leq \frac{\delta}{|4x|}.$$

We still have to get rid of the term $\frac{1}{|4x|}$ by seeing that it is smaller than some number, let's call it C for now. Intuitively, this term should not be a problem because we are interested in what happens when x is close to 4; and of course $\frac{1}{|4x|}$ only gets big when x gets close to zero. But x cannot be close to 0 and 4 at the same time when by "close" we mean something suitably near - so when it is close to 4, it is not close to 0, and we win, because we can then estimate

$$\frac{\delta}{|4x|} \leq C\delta$$

and this we can make smaller than ϵ simply by choosing any $\delta \leq \frac{\epsilon}{C}$.
This is probably the point where a lot of you got confused, so let us try to carefully rephrase the above argument in mathematical terms.
We want to bound $\frac{1}{|4x|}$ by some number. If we knew that the values of x that we have to consider (namely those x with $|x - 4| < \delta$) are bounded away from 0, by which we mean

there is a positive number b so that $|x| > b$ for all our values of x

then we could estimate

$$\frac{1}{|4x|} \leq \frac{1}{4b}$$

simply because $\frac{1}{|4x|}$ is biggest when $|x|$ is smallest; but we know that it never gets smaller than b.
Now, however, we can make mathematical sense of saying that x can not be close to 4 and 0 simultaneously. Namely, if x has distance less than 1 from 4, then the x closest to 0 is 3, or, in formulas,

$$|x-4| \leq 1 \text{ implies } |x| = |x-4+4| > |||x-4|-4|| \geq 3$$

(don't worry about this formal estimate too much - it is much easier to do the estimate by looking at the real line, for instance by drawing where x is allowed to be).
So if we make sure that in the end we pick our δ smaller than 1, we can from now on assume $|x| > 3$ and estimate

$$\frac{1}{|4x|} \leq \frac{1}{4*3} = \frac{1}{12},$$

so our number C from above would be $\frac{1}{12}$. Then we are basically done, but to realize this let's write down the whole chain of estimates we're doing:

$$|\frac{1}{x} - \frac{1}{4}| = \frac{|4-x|}{|4x|} \leq \frac{\delta}{|4x|} \leq \frac{\delta}{12},$$

where the first inequality is because we're only considering x that have $|x-4| < \delta$, the second one is caused by making sure that x is not close to zero and we now want to pick δ small enough so that the last term is smaller than ϵ. Additionally, since we have assumed that $\delta < 1$ for the bound on $\frac{1}{|4x|}$, we must make sure that the δ that we nominate isn't bigger than 1. This is easy, namely, we can choose

$$\delta = \min(1, \frac{\epsilon}{C}) = \min(1, 12\epsilon),$$

where $\min(a, b)$ is just the smaller of the numbers a, b. This choice makes sure that we can not only estimate $\frac{\delta}{12} \leq \epsilon$, but that we are at the same time allowed to use $\frac{1}{|4x|} \leq \frac{1}{12}$.

Remark: If you write down solutions like this, it is best to verify that the δ works. Try doing it for this case. If you nominate a δ, now matter how you obtained it, and you show that it works, you have written down a mathematically correct proof and deserve full credit (you could, however, get a penalty if you do illegal things in the derivation - but assuming that you make clear that the derivation is not supposed to be part of your solution, you're in the clear).
Remark: We could also have chosen $\delta = \min(1, \epsilon)$ because this looks nicer. Generally, you're always allowed to make δ smaller because then the condition $|f(x) - L| < \epsilon$ has to be checked for a smaller set of x, so it is still satisfied.
Remark Depending on the function that we're inspecting, what we win by forcing x to be close to a in the argument depends on which "unwanted" terms involving x come up during the estimate. In our case, it was of type $\frac{1}{|x|}$; the enemy then is getting close to zero. In other cases, you have to worry much less: For $f(x) = x^2$, for instance, you just need to know that the x we consider are bounded, e.g. $|x| < C$, for then the estimate goes like

$$|x^2 - L| = |x + \sqrt{L}||x - \sqrt{L}| \leq |x + \sqrt{L}|\delta \leq (|x| + \sqrt{L})\delta \leq (C + \sqrt{L})\delta,$$

and $(C + \sqrt{L})$ is just a fixed number in an actual computation (here $a = \sqrt{L}$).
Hence, in this case, assuming that δ is less than 1 (or any other fixed number) is enough, for then $|x| = |x - a + a| \leq |x - a| + |a| \leq 1 + |a|$.