Amalgamation and the finite model property

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The motivating phenomenon

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$\mathrm{Th}(G_R)$ has the *finite model property* (FMP): Every sentence in $\mathrm{Th}(G_R)$ has a finite model.

Moreover, this happens for a good probabilistic reason: A 0-1 law.
For each n, there is a natural probability measure μ_n on the structures in K of size n (the uniform measure, in this case) such that for all $\varphi \in \mathrm{Th}(G_R)$,

$$\lim_{n\to\infty} \mu_n(\{A \mid A \models \varphi\}) = 1.$$

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Cherlin's Question (Open)

Does the theory of the generic triangle-free graph have the FMP?

Two quotes

"When does a homogeneous structure for a finite relational language have the finite model property? More broadly, is there anything of interest in graph theory besides randomness and algebra?"
- Cherlin, *Exercises for logicians*

"In all those homogeneous structures which I know to have the finite model property, [it] arises either from probabilistic arguments as above [0-1 laws], or from stability, or conceivably from a mixture of these."
- Macpherson, *A survey of homogeneous structures*

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Idea: Rule out algebra/stability and show that in the remaining "purely combinatorial" examples, the finite model property is always explained by randomness/probability.

Since the generic triangle-free graph seems to be "purely combinatorial", a realization of this idea should answer Cherlin's Question negatively.

The model-theoretic setting

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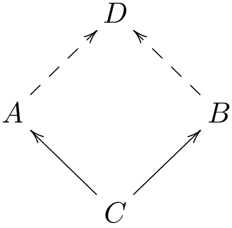
These theories arise as Fraïssé limits of Fraïssé classes in relational languages with *disjoint amalgamation*: Given an amalgamation diagram



we can choose D and embeddings $A \to D$ and $B \to D$ in such a way that the intersection of the images of A and B in D equals the image of C.

n-amalgamation

Definition

A complete type p is *non-redundant* if it does not contain the formula $x = y$ for any distinct variables x and y.

Definition

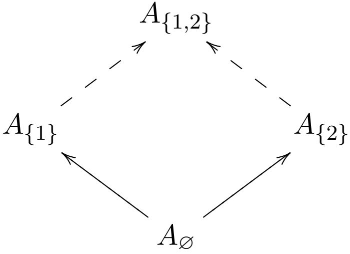
Let $x_1, \ldots, x_n$ be tuples of distinct variables. Given $S \subseteq [n]$, let X_S be the variable context $\{x_i\}_{i \in S}$. An *n-amalgamation problem* (over A) is given by a non-redundant type p_S (over A) in the variable context X_S for each $S \subseteq [n]$ with $|S| = n - 1$, such that $p_S \restriction X_{S \cap T} = p_T \restriction X_{S \cap T}$ for all $S \neq T$. A *solution* to the n-amalgamation problem is a non-redundant type $p_{[n]}$ in the variable context $X_{[n]}$ such that $p_{[n]} \restriction X_S = p_S$ for all S.

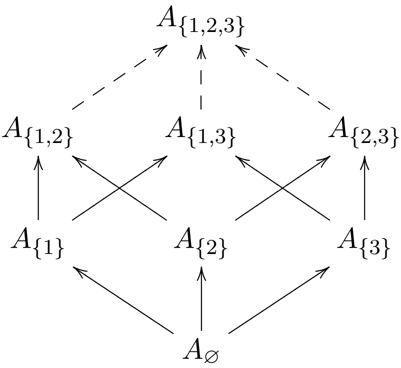
Definition

T has *n-amalgamation* if every n-amalgamation problem has a solution.

n-amalgamation

For theories arising from Fraïssé limits, n-amalgamation for the theory is equivalent to (disjoint) n-amalgamation for the Fraïssé class. Here are pictures of 2-amalgamation and 3-amalgamation:





Note that 2-amalgamation is just the disjoint amalgamation property. Indeed, every $\aleph_0$-categorical theory with trivial acl has 2-amalgamation.

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Sketch of proof: Morleyize T, so it has an $\forall\exists$ axiomatization.
For all N, we describe a probability measure μ_N on structures with domain $[N]$, according to the following inductive probabalistic construction:

- For $i \in [N]$, pick the 1-type of $\{i\}$ uniformly at random from $S^1(T)$.
- Suppose we have assigned non-redundant n-types from $S^n(T)$ to all subsets of $[N]$ of size n. Given $X \subseteq [N]$ of size $n+1$, choose a non-redundant type from $S^{n+1}(T)$ uniformly at random from those amalgamating the n-types assigned to the subsets of X of size n.

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A computation shows that for any finite collection of the $\forall\exists$ axioms of T,

$$\lim_{N\to\infty} \mu_N(\{A \mid A \models \bigwedge_{i=1}^{k} \varphi_i\}) = 1.$$

Does the converse hold?

Naive Conjecture (Version 1)

An $\aleph_0$-categorical theory with trivial acl has the finite model property if and only if it has n-amalgamation for all n.

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Problem: Equivalence relations.

T^*_{feq}

Let L be the language with two sorts, O and P, and one relation $E_x(y, z)$ in the variables x of sort P and y, z of sort O.

Let K be the class of finite L-structures such that for all a of sort P, E_a is an equivalence relation on sort O. K is a Fraïssé class with disjoint amalgamation, and T^*_{feq} is the theory of its Fraïssé limit.

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Solution: Just rule out equivalence relations.

Primitive combinatorial theories

Definition

A *primitive combinatorial theory* is a complete $\aleph_0$-categorical theory with trivial acl such that for any finite set A and any complete 1-type p over A, there are no nontrivial A-definable equivalence relations on the realizations of p.

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There are examples of primitive combinatorial theories with the FMP which fail n-amalgamation for some n. But every example I know is a reduct of a primitive combinatorial theory with n-amalgamation for all n.

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Solution: Adjust the conjecture.

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A primitive combinatorial theory has the finite model property if and only if it is a reduct of a primitive combinatorial theory with n-amalgamation for all n.

Despite the fact that it quantifies over all primitive combinatorial expansions of a theory T, this conjecture (if true) would be a useful test for the finite model property.

Theorem

The theory of the generic triangle-free graph is primitive combinatorial, and no primitive combinatorial expansion of it has 3-amalgamation.

Stable primitive combinatorial theories

Theorem (Macpherson)

Let T be a theory with trivial acl, and suppose that a stable formula defines an infinite and coinfinite subset of $M \models T$. Then there is a nontrivial 0-definable equivalence relation on the domain of M.

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Theorem (Macpherson)

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Corollary

Let T be a primitive combinatorial theory with complete 1-types isolated by formulas $\{\theta_i\}_{i=1}^m$, and let $\varphi(x, \overline{y})$ be a stable formula. For any $\overline{b}$, $\varphi(x, \overline{b})$ is equivalent to a boolean combination of $\theta_i(x)$ and $x = b_j$.

Corollary

Every stable primitive combinatorial theory is interdefinable with the theory of n infinite partitioning unary predicates for some n.

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So the primitive combinatorial theory notion effectively rules out nontrivial stable behavior.

NIP primitive combinatorial theories

Theorem

Let T be a primitive combinatorial theory. If T is finitely axiomatizable, then T is distal. In particular, T is NIP and unstable. Conversely, if T is distal and the language is finite, then T is finitely axiomatizable.

The proof uses the finitary "strong honest definitions" characterization of distality given by Chernikov and Simon.

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In keeping with the philosophy that distal theories are the "purely unstable" NIP theories:

Conjecture

Every unstable NIP primitive combinatorial theory is distal.

Simple primitive combinatorial theories

Theorem

Let T be a primitive combinatorial theory. The following are equivalent:

1. T has 3-amalgamation.
2. T has trivial forking: $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ if and only if $A \cap B \subseteq C$.
3. T is supersimple of U-rank 1.

If the stable forking conjecture is true for primitive combinatorial theories, these are equivalent to:

4. T is simple.

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Conjecture

If a primitive combinatorial theory is not simple, then it has SOP3.

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Thank you!