

Unboundedness of some higher Euler classes

Kathryn Mann

Abstract

In this paper, we study Euler classes in groups of homeomorphisms of Seifert fibered 3-manifolds. We show that, in contrast to the familiar Euler class for $\text{Homeo}_0(S^1)^\delta$, these Euler classes for $\text{Homeo}_0((M^3)^\delta)$ are *unbounded* classes. In fact, we give examples of flat topological M bundles over a genus 3 surface whose Euler class takes arbitrary values.

For a topological group G , let G^δ denote G with the discrete topology, and $H^*(G^\delta; R)$ the (group) cohomology of G^δ with $R = \mathbb{Z}$ or \mathbb{R} coefficients. When G is the group of homeomorphisms or diffeomorphisms of a manifold M , elements of $H^*(G^\delta; R)$ are characteristic classes of flat or foliated M bundles with structure group G . One says that a class is *bounded* if it has a cocycle representative taking a bounded set of values. Determining which classes are bounded is an interesting and often very difficult question in its own right (see [12] for an introduction to this and related problems) but particularly relevant in the case where G is a subgroup of $\text{Homeo}(M)$ for some M . In this case, bounds on characteristic classes give obstructions for topological M -bundles to be flat. On the flipside, showing that a class has *no* bounded representative is also an interesting as it often amounts to constructing interesting examples of flat bundles.

The earliest example of a bounded class comes from Milnor [18], who showed the Euler class for $\text{SL}(2, \mathbb{R})^\delta$ is bounded. Wood [20] generalized this argument to $\text{Homeo}_0(S^1)$, the identity component of $\text{Homeo}(S^1)$, (which naturally contains $\text{SL}(2, \mathbb{R})$ as a subgroup) to obtain a complete characterization of the oriented, topological circle bundles over surfaces that admit a foliation transverse to the fibers¹. In modern language, their work can be reframed as follows:

Milnor–Wood inequality [18, 20]. The real Euler class in $H^2(\text{Homeo}_0(S^1)^\delta; \mathbb{R})$ is bounded, and has (Gromov) norm equal to $1/2$.

More generally, when G is a real algebraic subgroup of $\text{GL}(n, \mathbb{R})$, it follows from [9] that elements of $H^*(G^\delta; \mathbb{R})$ have bounded representatives, and explicit bounds on their norms have been computed in several cases. See eg. [3, 6, 7] and references therein. However, much less is known for homeomorphism groups. Following an easy argument of Anderson [1] and a hard result of Edwards–Kirby [8], we know that $H^1(\text{Homeo}_0(M)^\delta; \mathbb{Z}) = 0$ for

¹In the smooth setting, this is equivalent to admitting a flat connection, hence, even in the topological case such bundles are called “flat.”

any compact manifold M , so the first nontrivial examples of characteristic classes for flat bundles arise in degree two. Besides the aforementioned $M = S^1$, the most basic example of this is $M = \mathbb{R}^2$. (Despite noncompactness, $H^1(\text{Homeo}_0(\mathbb{R}^2))$ is also zero.) Calegari [4] demonstrated that the Euler class in $H^2(\text{Homeo}_0(\mathbb{R}^2)^\delta; \mathbb{R})$ – and in fact also its pullback to $H^2(\text{Diff}_0(\mathbb{R}^2)^\delta; \mathbb{R})$ – is unbounded. But beyond this essentially nothing is known.

Here we consider Euler classes in homeomorphism groups of 3-manifolds. The existence of such classes comes from the work of Hatcher, Ivanov, and McCullough and Soma [10, 14, 17] who prove that, for many closed, prime Seifert fibered 3-manifolds M , rotation of the fibers gives a homotopy equivalence $\text{SO}(2) \rightarrow \text{Homeo}_0(M)$. Together with a deep result of Thurston, this implies that $H^*(\text{Homeo}_0(M)^\delta; \mathbb{Z})$ is generated by an “Euler class” in degree 2. In a few other cases, including the obvious $M = T^2$, but also $M = T^3$ and some lens spaces, the inclusion $\text{SO}(2) \rightarrow \text{Homeo}_0(M)$ obtained by rotating fibers of a fibration or Seifert fibration induces an inclusion $\pi_1(\text{SO}(2)) \rightarrow \pi_1(\text{Homeo}_0(M))$ as a factor in a direct product decomposition, giving classes in $H^2(\text{Homeo}_0(M)^\delta; \mathbb{Z})$ that are also analogous to the Euler class for $\text{Homeo}_0(S^1)$. These are described in more detail in §2.1.

Our main result is that *all* of these Euler classes are unbounded. Precisely, we show:

Theorem 1.1. Let M be a closed Seifert fibered 3-manifold where rotation of fibers gives a homotopy equivalence $\text{SO}(2) \hookrightarrow \text{Homeo}_0(M)$. Then the Euler class in $H^2(\text{Homeo}_0(M); \mathbb{Z})$ is unbounded. More generally, if M is such that the inclusion $\text{SO}(2) \hookrightarrow \text{Homeo}_0(M)$ induces an inclusion of $\pi_1(\text{SO}(2))$ as a direct factor in $\pi_1(\text{Homeo}_0(M))$, then any class $\alpha \in H^2(\text{Homeo}_0(M); \mathbb{R})$ with nonzero image in $H^2(\text{SO}(2); \mathbb{R})$ is unbounded.

This is a direct consequence of the following stronger result.

Theorem 1.2. Let M be as in the general case of Theorem 1.1, and let $e \in H^2(\text{Homeo}_0(M); \mathbb{Z})$ have nonzero image in $H^2(\text{SO}(2); \mathbb{Z})$. Then, for any k , there exists a homomorphism ρ from the fundamental group of a genus 3 surface Σ to $\text{Homeo}_0(M)$ such that $\langle \rho^*(e), [\Sigma] \rangle = k$.

Our proof is fundamentally different than Calegari’s proof of unboundedness of the Euler class for $\text{Homeo}_0(\mathbb{R}^2)^\delta$, which uses non-compactness of \mathbb{R}^2 in an essential way. It also differs considerably from an existing argument for unboundedness of cohomology classes in $\text{Homeo}_0(T^2)$ (see discussion in Section 2.1), which used the fact that $H^2(\text{Homeo}_0(T^2); \mathbb{Z}) \cong \mathbb{Z}^2$ has a $\text{GL}(2, \mathbb{Z})$ action (via conjugation, using the mapping class group of T^2).

Section 2 contains some brief background on bounded cohomology, Gromov norm, and cohomology of homeomorphism groups, giving the tools to derive Theorem 1.1 from Theorem 1.2. The proof of Theorem 1.2 is an explicit construction described in Section 3.

Measure-preserving homeomorphisms. By contrast, suppose that μ is a probability measure on M and let G_μ denote the subgroup of measure-preserving homeomorphisms in $\text{Homeo}_0(M)$. In contrast to Theorem 1.1, in the measure-preserving case we have the following.

Theorem 1.3. Let M be as in Theorem 1.1. Let $e \in H^2(\text{Homeo}_0(M)^\delta; \mathbb{Z})$ be a class with nonzero image in $H^2(\text{SO}(2); \mathbb{R})$. Then the pullback of e to $H^2(G_\mu^\delta; \mathbb{Z})$ is zero.

By averaging a measure over the $\mathrm{SO}(2)$ action, one may assume that μ is invariant under rotation of fibers, so there is an inclusion $\mathrm{SO}(2) \rightarrow G_\mu \rightarrow \mathrm{Homeo}_0(M)$. In this case, Theorem 1.3 implies that the Euler class in $H^2(G_\mu^\delta; \mathbb{Z})$ is zero.

In particular, for the special case of the 2-dimensional torus, since $\pi_1(T^2) = \mathbb{Z}^2$ is amenable (so any action on a manifold M has an invariant measure), this gives

Corollary 1.4. For M as in Theorem 1.3, flat M -bundles over T^2 always have zero Euler class.

Note that this statement would be implied by boundedness of $e \in H^2(\mathrm{Homeo}_0(M)^\delta; \mathbb{Z})$.

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2 Preliminaries

We quickly review the standard theory of bounded cohomology and set up notation. A reader who is well-acquainted with bounded cohomology can skip quickly to Section 2.1 where we discuss cohomology of homeomorphism groups.

For M a manifold, and $a \in H_*(M; \mathbb{R})$ an element of singular homology, define a pseudonorm

$$\|a\| := \inf\{\sum |c_i| : [\sum c_i \sigma_i] = a\}$$

where the infimum is taken over all real singular chains representing a in homology. The L_1 norm on singular chains used in this definition gives a dual L_∞ norm on singular cochains; and the set of bounded cochains forms a subcomplex of $C^*(M)$. The cohomology of this complex is the *bounded cohomology* $H_b^*(M; \mathbb{R})$ of M . The (pseudo-)norm, $\|\alpha\|$, of a cohomology class α is the infimum of the L_∞ norms of representative cycles; and if $\|\alpha\|$ is finite we say that it is a *bounded class*.

One can extend these definitions quite naturally to the Eilenberg–MacLane group cohomology. Recall that, for a discrete group G , the set of *inhomogeneous k -chains*, $C_k(G)$, is the free abelian group generated by k -tuples $(g_1, \dots, g_k) \in G^k$ with an appropriate boundary operator. The homology of this complex is the (integral) group homology $H_k(G; \mathbb{Z})$; and $H_k(G; \mathbb{R})$ is the homology of the complex $C_*(G) \otimes \mathbb{R}$. The homology of the dual complexes $\mathrm{Hom}(C_k, \mathbb{Z})$ and $\mathrm{Hom}(C_k, \mathbb{R})$ give the *group cohomology* $H^k(G; \mathbb{Z})$ and $H^k(G; \mathbb{R})$ respectively. As in the singular homology case above, there is a natural L_1 norm on k -chains given by $\|\sum s_i(g_{i,1}, \dots, g_{i,k})\| = \sum |s_i|$, which descends to a pseudonorm on homology by taking the infimum over representative cycles. We also have a dual L_∞ norm on $C^k(G)$, and for $\alpha \in H^*(G; \mathbb{R})$ define

$$\|\alpha\| := \inf\{\|c\|_\infty : [c] = \alpha\}.$$

Again, bounded (co)cycles are those with finite norm. Note that $\|\alpha\|$ is finite if and only if there exists D such that $|\alpha(g_1, g_2, \dots, g_k)| < D$ holds for all $(g_1, g_2, \dots, g_k) \in G^k$.

A remarkable theorem of Gromov allows one to pass between groups and spaces:

Theorem 2.1 ([9]). There is a natural isometric isomorphism $H_b^*(\pi_1(M); \mathbb{R}) \rightarrow H_b^*(M; \mathbb{R})$.

We will make use of this in the next subsection.

Computing norms. In degree two, there is an effective means of computing the norm of a cohomology class through representations of surface groups. An integral class $c \in H_2(G; \mathbb{Z})$ can always be realized as the image of a map from a closed orientable surface Σ of genus ≥ 1 into a $K(G, 1)$ space; such a map induces a homomorphism $\rho : \pi_1(\Sigma) \rightarrow G$. Thus, on the level of group cohomology we have $c = \rho_*([\Sigma])$ and

$$\langle \alpha, c \rangle = \langle \rho^*(\alpha), [\Sigma] \rangle.$$

It is easy to verify that $[\Sigma]$ has norm $2\chi(\Sigma)$ (See [9, §2] for the computation.) Hence, $\|c\| \leq 2\chi(\Sigma)$, and showing that α is an *unbounded* class amounts to showing that

$$\sup_{\rho: \pi_1(\Sigma) \rightarrow G} \frac{\langle \rho^*(\alpha), [\Sigma] \rangle}{2\chi(\Sigma)} = \infty,$$

where the supremum is taken over all homomorphisms from surface groups into G .

Integrally, the quantity $\langle \rho^*(\alpha), [\Sigma] \rangle$ can be easily read off from a central extension. There is a well known correspondence between $H^2(G; A)$ and central extensions of G by A for any abelian group A . If $\alpha \in H^2(G; \mathbb{Z})$ is represented by the extension $0 \rightarrow \mathbb{Z} \rightarrow \hat{G} \rightarrow G \rightarrow 1$, then $\rho^*(\alpha)$ is represented by the pullback $0 \rightarrow \mathbb{Z} \rightarrow \rho^*(\hat{G}) \rightarrow \pi_1(\Sigma) \rightarrow 1$.

For a surface Σ of genus g , we have a standard presentation

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] \rangle$$

In this case, the integer $\langle \rho^*(\alpha), [\Sigma] \rangle$ can be computed by taking lifts \tilde{a}_i, \tilde{b}_i of the generators a_i and b_i to elements of $\rho^*(\hat{G})$. Since this is a central extension, the value of any commutator $[\tilde{a}_i, \tilde{b}_i]$ is independent of the choice of lifts \tilde{a}_i and \tilde{b}_i . The product of commutators $\prod_{i=1}^g [\tilde{a}_i, \tilde{b}_i]$ projects to the identity in $\pi_1(\Sigma)$, so can be identified with an element $n \in \mathbb{Z}$. In this case, one checks easily from the definition that $n = \langle \rho^*(\alpha), [\Sigma] \rangle$.

We note that, although not framed in the language of bounded cohomology, this strategy for computation is already present in Milnor and Wood's work in [18] and [20] respectively.

2.1 Euler classes of homeomorphism groups

This section describes the known analogs of the Euler class in $\text{Homeo}_0(M)$, for various manifolds M , justifying some of the remarks made in the introduction. Our starting point is the following remarkable theorem of Thurston.

Theorem 2.2 (Thurston [19]). Let M be a differentiable manifold. The identity homomorphism $\text{Homeo}(M)^\delta \rightarrow \text{Homeo}(M)$ induces an isomorphism $H^*(\text{Homeo}(M)^\delta; \mathbb{Z}) \cong H^*(B\text{Homeo}(M); \mathbb{Z})$.

Here we focus on the induced isomorphism on cohomology of the identity components $\text{Homeo}_0(M)^\delta \rightarrow \text{Homeo}_0(M)$.

As a sample application of this theorem, one can conclude that the Euler class and its powers are the only characteristic classes of flat topological circle bundles. To see this, one first shows that the inclusion of the group of rotations $\text{SO}(2)$ into $\text{Homeo}_0(S^1)$ is a homotopy equivalence (a relatively easy exercise), hence $H^*(\text{Homeo}_0^\delta(S^1); \mathbb{Z}) \cong H^*(B\text{SO}(2); \mathbb{Z})$. As is well known, $B\text{SO}(2) \cong \mathbb{C}P^\infty$ is generated by the Euler class in degree two.

To apply Thurston's theorem in other situations, we look for other manifolds where the homotopy type (or at least the cohomology) of $\text{Homeo}_0(M)$ is known. In dimension 2, as was mentioned in the introduction, $S^1 \rightarrow \mathbb{R}^2$ is a homotopy equivalence, but in contrast to the $M = S^1$ case, the Euler class of $\text{Homeo}_0(\mathbb{R}^2)^\delta$ is unbounded by [4]. For $M = T^2 = S^1 \times S^1$, rotating either of the S^1 factors gives a continuous homomorphism $\text{SO}(2) \rightarrow \text{Homeo}_0(T^2)$ which is injective on π_1 – in fact, the inclusion $\text{SO}(2) \times \text{SO}(2) \rightarrow \text{Homeo}_0(T^2)$ is a homotopy equivalence. Thus, the pullback of the Euler class in $H^2(B\text{SO}(2); \mathbb{Z})$ to $H^2(B\text{Homeo}(T^2); \mathbb{Z}) \cong H^2(\text{Homeo}(T^2)^\delta; \mathbb{Z})$ by either of these inclusions is nontrivial. A direct computation, given in [16, §4.2], shows that these classes are also unbounded.

Seifert fibered 3-manifolds give the natural generalization of the examples above to dimension 3, as the fibering gives an $\text{SO}(2)$ subgroup of $\text{Homeo}_0(M)$ acting freely on M . These are essentially the only other examples where the homotopy type of $\text{Homeo}_0(M)$ is both known and known to have a homotopically nontrivial $\text{SO}(2)$ subgroup.

The Haken case is covered by the following theorem of Hatcher and Ivanov.

Theorem 2.3 ([10], [14]). Suppose M is an orientable, Haken, Seifert-fibered 3-manifold, $M \neq T^3$. Then the inclusion $S^1 \rightarrow \text{Homeo}_0(M)$ by rotations of the fibers is a homotopy equivalence.

In the case of T^3 , it is also known that $\text{Homeo}_0(T^3) \cong T^3$. We remark that Hatcher's original proof was in the PL category, but (as noted by Hatcher) this is equivalent to the topological category by the triangulation theorems of Bing and Moise [2, 11]. Ivanov's proof of the theorem above is for groups of diffeomorphisms, but an argument due to Cerf, together with Hatcher's later proof of the Smale conjecture implies that the inclusion of $\text{Diff}(M^3)$ into $\text{Homeo}(M^3)$ is a homotopy equivalence; this makes the smooth category equivalent as well.

McCullough–Soma [17] proved $\widetilde{\text{Homeo}_0(M)} \cong S^1$ for the small Seifert-fibered non-Haken manifolds with $\mathbb{H}^2 \times \mathbb{R}$ and $SL(2, \mathbb{R})$ geometries. For spherical manifolds, (and for prime 3-manifolds in general) it is conjectured that the inclusion $\text{Isom}(M) \rightarrow \text{Homeo}(M)$ is a homotopy equivalence, and this is known to be true in most cases by work of Ivanov [15] and later Hong, Kalliongis, McCullough and Rubinstein. See [13] for references and a detailed exposition, as well as a table of homotopy types of $\text{Isom}(M)$ for the known cases. In several of these, rotation of the fibers gives a homotopically nontrivial $\text{SO}(2)$ subgroup which is a factor of π_1 , hence examples to which Theorem 1.1 applies.

3 Proof of Theorems 1.1 and 1.2

Let M be a Seifert fibered 3-manifold, and let $G = \text{Homeo}_0(M)$. Let $\iota : \text{SO}(2) \rightarrow G$ be the action of rotating the fibers, and suppose that ι induces an inclusion $\mathbb{Z} \cong \pi_1(\text{SO}(2)) \rightarrow \pi_1(G)$ as a factor in a splitting as a direct product. Let \tilde{G} be the covering group of G corresponding to the subgroup $\pi_1(G)/\iota(\mathbb{Z}) \subset \pi_1(G)$. (Recall that G is locally contractible by Cernavskii [5] or Edwards–Kirby [8], so standard covering space theory applies here.) If ι is also surjective on π_1 , for instance, a homotopy equivalence, then \tilde{G} is the universal covering group of G . In general, it is a central extension $0 \rightarrow \mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$.

We will show that this central extension represents a class e in $H^2(\text{Homeo}_0(M)^\delta; \mathbb{Z}) \cong H^2(\text{Homeo}_0(M); \mathbb{Z}) \cong \mathbb{Z}$ that is *unbounded*. This will prove Theorem 1.1. Following the framework discussed in Section 2, to show that e is unbounded, it suffices to construct representations of surface groups $\rho : \pi_1(\Sigma) \rightarrow \text{Homeo}_0(M)$ with $\rho^*(e)/\chi(\Sigma)$ arbitrarily large. Although, in using this strategy, *a priori* one may need to vary the genus of surface to construct representations with increasingly large values of $\rho^*(e)/\chi(\Sigma)$, in this case we need only to work with a surface of genus 3.

Put otherwise, we will show how to construct commutators $[a_i, b_i]$ with a_i and $b_i \in G$ (for $i = 1, 2, 3$), such that $\prod_{i=1}^3 [a_i, b_i] = \text{id}$, but lifts $\prod_{i=1}^3 [\tilde{a}_i, \tilde{b}_i]$ to \tilde{G} represent unbounded covering transformations. This will prove Theorem 1.2.

The first step is a local construction of bump functions.

Definition 3.1. A *standard bump function* on D^2 is a function $D^2 \rightarrow \mathbb{R}$, which, after conjugation by some $h \in \text{Homeo}_0(D^2)$ agrees with

$$f(re^{i\theta}) = \begin{cases} 1 & \text{if } r < 1/3 \\ 2 - 3r & \text{if } 1/3 \leq r \leq 2/3 \\ 0 & \text{if } r > 2/3 \end{cases}$$

What we have in mind as particular examples are piecewise linear (or piecewise smooth) functions $f : D^2 \cong [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}$ that are identically 0 on a neighborhood of the boundary, identically 1 on a neighborhood of $(0, 0)$, and with the level sets $f^{-1}(p)$ for $p \in (0, 1)$ piecewise linear (or piecewise smooth) curves. Moreover, these should have the property that some line λ from 0 to $\partial([-1, 1] \times [-1, 1])$ is transverse to each level set of f , with f monotone along λ . In this case, one can easily construct the conjugacy h to the function above defined on the round disc as follows. For $p \in (0, 1)$, let ℓ_p be the total arc length of $f^{-1}(p)$ and, for $x \in f^{-1}(p)$, let $\ell_p(x)$ denote the arc length of the segment of $f^{-1}(p)$ (oriented as the boundary of $f^{-1}([p, 1])$) from $\lambda \cap f^{-1}(p)$ to x . Then, for $x \in f^{-1}(p)$, set $h(x) = \frac{2-p}{3} e^{i\ell_p(x)/\ell(p)}$. One may then extend h arbitrarily to a homeomorphism defined on $f^{-1}(0)$ and $f^{-1}(1)$.

Lemma 3.2. Let $T = D^2 \times S^1$ be a (p, q) standard fibered torus, let f be a standard bump function, and let $k \in \mathbb{R}$. There exist $a, b \in \text{Diff}(T)$ such that the commutator $[a, b]$ preserves fibers and rotates the fiber $\{x\} \times S^1$ by $2\pi k f(x)$ if $x \neq 0$, and the exceptional fiber by $2\pi qk$.

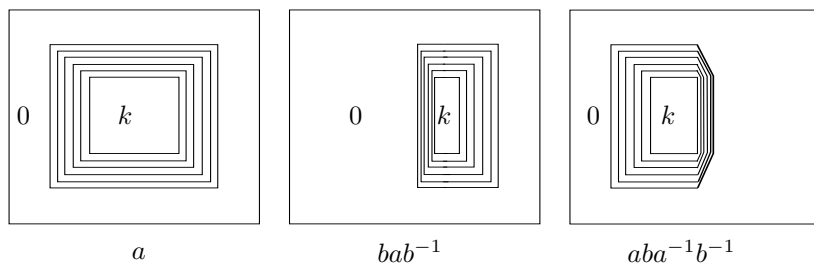


Figure 1: Level sets of PL bump functions

Proof. We take local coordinates to identify D^2 with the rectangle $[-3, 3] \times [-3, 3] \subset \mathbb{R}^2$, so that the exceptional fiber passes through $(0, 0)$, and we work in the PL setting. First, define ϕ to be a standard bump function that is 1 on $[-1, 1]^2$, zero on the complement of $[-2, 2]^2$, and in the topological annulus between these regions of definition, it is linear on each of the four sets cut out by the diagonals of $[-3, 3]^2$. Level sets of a are shown in Figure 1 left. For a point (x, s) in $[-3, 3]^2 \times S^1$, define $a(x, s) = (x, s + 2\pi qk\phi(x))$ if $x \neq (0, 0)$ (i.e. a rotation of the fiber over x by $2\pi qk\phi(x)$), and define a to be a rotation by $2\pi k$ on the exceptional fiber.

To construct b , first define $F : [-3, 3] \rightarrow [-3, 3]$ by

$$F(u) = \begin{cases} u & \text{if } u \geq 1 \\ \frac{u+2}{3} & \text{if } -2 < u < 1 \\ 3(u+2) & \text{if } -3 \leq u \leq -2 \end{cases}$$

and define b on $[-3, 3] \times [-3, 3] \times S^1$ by $b(u, v, e^{i\theta}) = (F(u), v, e^{i\theta})$.

Since both a and b preserve fibers, $ba^{-1}b^{-1}$ does as well. Moreover, $ba^{-1}b^{-1}$ rotates the fiber through a point $x \in [-3, 3]^2$ by $-qk\phi(b^{-1}(x))$ for $x \neq 0$, and by $k\phi(b^{-1}(x))$ on exceptional fiber. It is now easily verified that the composition $aba^{-1}b^{-1}$ rotates the fibers over x by the standard bump function with level sets depicted in Figure 1 right. \square

The next step is to glue the bump functions given by Lemma 3.2 together into a nice partition of unity, subordinate to an open cover consisting of only three sets.

Lemma 3.3. Let S be an orientable topological surface. There exists an open cover $\mathcal{O} = \{O_1, O_2, O_3\}$ of S , with each O_i a union of disjoint homeomorphic open balls, and a partition of unity λ_i subordinate to \mathcal{O} such that the restriction of λ_i to any connected component of O_i is a standard bump function.

Proof. Let $\Gamma = (V, E)$ be a degree three graph on S , with polygonal faces. For example, Γ may be constructed as the dual graph to a triangulation of S . First we define $\mathcal{O} = \{O_1, O_2, O_3\}$. Let N_δ denote the union of the δ -neighborhoods of the edges in Γ . Fixing an appropriate metric and PL structure on S , we may assume that the boundary of N_δ , for any sufficiently small $\delta > 0$, consists of line segments parallel to the edges of Γ .

Fixing δ , let $O_1 = S \setminus N_{\delta/2}$. Choose δ small enough so that connected components of O_1 are in one to one correspondence with faces of the graph, each the complement of a

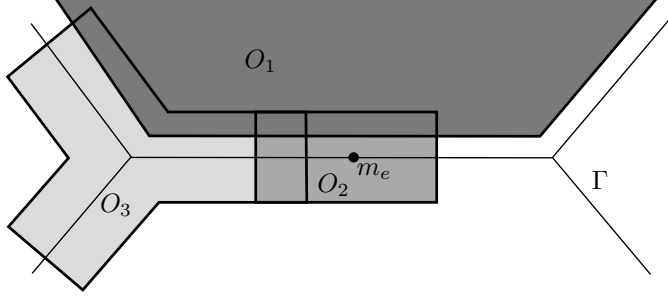


Figure 2: A cover supporting a good partition

small $\delta/2$ -neighborhood of the boundary of the face. For each edge e , let m_e denote its midpoint. In a neighborhood of m_e , N_δ has natural local coordinates as $(-\delta, \delta) \times (-1, 1)$ with the edge given by $0 \times (-1, 1)$, $m_e = (0, 0)$ and lines $\{p\} \times (-1, 1)$ parallel to the edge. We assume that δ is small enough so that we may choose these neighborhoods of midpoints to be pairwise disjoint and let U_e denote the neighborhood containing m_e . Let O_2 be the union $\bigcup_{e \in \text{edge}} U_e$. Finally, let X be the union of the sub-neighborhoods $(-\delta, \delta) \times [-1/2, 1/2]$ and let O_3 be the complement of X in N_δ . See figure 2 for a local picture.

We now construct the desired partition of unity, with λ_i supported on O_i . Define λ_1 to be constant 1 on $S \setminus N_\delta$, constant 0 on $N_{\delta/2}$, and piecewise linear in the intermediate regions, with level sets consisting of polygons with edges parallel to the edges of Γ .

Let $g = 1 - \lambda_1$, this is a function supported on $O_2 \cup O_3$. Define λ_2 to agree with g on the complement of $\bigcup_{e \in E} U_e$. In coordinates $U_e = (-\delta, \delta) \times (-1, 1)$ as given above, define the restriction of λ_2 to U_e to agree with g on $(-\delta, \delta) \times (-1/2, 1/2)$, to be given by $\lambda_2(x, y) = 2(1 - |y|)g(x, y)$ on $(-\delta, \delta) \times (-1, -1/2) \cup (-\delta, \delta) \times (1/2, 1)$, and then extend λ_2 to be 0 elsewhere. This gives a continuous (in fact, piecewise linear) bump function supported on O_2 . Finally, let $\lambda_3 = 1 - \lambda_1 - \lambda_2$, which is supported on O_3 . It is easily verified that this is a standard bump function, as in the example discussed after Definition 3.1. \square

To finish the proof of Theorem 1.2, let M be a Seifert fibered 3-manifold, and let S be the base orbifold. Take a cover $\mathcal{O} = \{O_1, O_2, O_3\}$ of S as given by Lemma 3.3. Using the construction from Lemma 3.3 starting with a graph on M , we may arrange for each exceptional fiber to be contained in only one set in \mathcal{O} , and also to have each connected component of each element of \mathcal{O} contain at most one exceptional fiber. Let $\{\lambda_i\}$ be the partition of unity subordinate to this cover consisting of standard bump functions.

Fix a connected component B of some set $O_i \in \mathcal{O}$, and let $B \times S^1$ be the union of fibers over B . By construction this is a (p, q) standard fibered torus for some p, q . Fix $K \in \mathbb{Z}$. Lemma 3.2 constructs homeomorphisms $a_B, b_B \in \text{Homeo}_0(M^3)$ supported on $B \times S^1$ such that $[a_B, b_B]$ rotates each (nonexceptional) fiber over $\{x\} \times S^1$ by $2\pi K \lambda_i(x)$. There is a natural path $a_B(t)$ from the identity in $\text{Homeo}_0(M)$ to $a_B(1) = a_B$ by applying the construction of Lemma 3.2 to give rotations of a (non-exceptional) fiber through x by $2\pi qt K \lambda_i(x)$ at time t .

Then $[a_B(t), b_B]$ gives a path from identity to $[a_B, b_B]$ that rotates (non-exceptional)

fibers by $2\pi qtK\lambda(x)$ at time t . Moreover, if $b_B(t)$ is any path from b_B to the identity supported on B , then $[a_B(t), b_B]$ is homotopic rel endpoints to $[a_B(t), b_B(t)]$. Let

$$a_i = \prod_B a_B, \text{ and } a_i(t) = \prod_B a_B(t)$$

where the product is taken over all connected components of O_i . Similarly, let

$$b_i = \prod_B b_B, \text{ and } b_i(t) = \prod_B b_B(t).$$

Let \tilde{G} be the covering group of $G = \text{Homeo}_0(M)$ as given at the beginning of this section; i.e. the central extension $0 \rightarrow \mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$. One definition of this covering group is as the set of equivalence classes of paths based at the identity in G , where two paths are equivalent if they have the same endpoint and their union is an element of $\pi_1(G)$ that belongs to the subgroup $\pi_1(G)/\iota(\mathbb{Z})$. The group operation is pointwise multiplication, or equivalently, concatenation. In this interpretation, the inclusion of $n \in \mathbb{Z}$ into \tilde{G} is given by a path g_t in G , $t \in [0, 1]$ that rotates (nonexceptional) fibers by an angle of $2\pi nt$ at time t .

Now we return to the machinery of Section 2. Consider the map of a genus 3 surface group into G where the images of the standard generators are a_i and b_i as defined above. The paths $a_i(t)$ and $b_i(t)$ give lifts of a_i and b_i to \tilde{G} , with commutator $[a_i(t), b_i(t)]$ a path from the identity to a map that rotates fibers by $2\pi K\lambda_i(x)$. Hence, $\prod_{i=1}^3 [a_i(t), b_i(t)]$ represents $K \in \mathbb{Z}$. Thus, if ρ is the associated map of the surface group, and e the Euler class in $H^2(G, \mathbb{Z})$, this means that $\langle \rho_*(\Sigma), e \rangle = K$. Since K can be chosen arbitrarily, this proves Theorem 1.2. \square

Remark 3.4. The constructions above can likely be realized in the smooth category (i.e. with a homomorphism $\pi_1(\Sigma_3) \rightarrow \text{Diff}_0(M)$), however, more care would be needed in the construction of the bump functions, as not all convex, smooth bump functions on a disc are smoothly conjugate.

4 Measure preserving case

We introduce some preliminary definitions. Let M be a Seifert fibered 3-manifold, and let ν denote the signed length measure on fibers, where regular fibers are normalized to have length one. Locally, in a standard (p, q) fibered torus $D^2 \times S^1$, where S^1 has unit length, ν is $\frac{1}{q}d\theta$. Define the fiber length of a rectifiable curve $\gamma : [0, 1] \rightarrow M$ by $\ell(\gamma) = \int_\gamma d\nu$. Note that this definition is independent of the homotopy class of γ rel endpoints. Thus, we can extend this definition to *continuous curves* by defining $\ell(\gamma)$ to be the length of a smooth approximation of γ that is homotopic to γ rel endpoints.

Let μ be a probability measure on M . For a path f_t in $\text{Homeo}_0(M)$ from $f_0 = \text{id}$ to $f_1 = f$, and point $x \in M$; $f_t(x)$ defines a continuous path in M . Thus, we may define the *average fiber rotation* with respect to μ by

$$R_\mu(f_t) := \int_M \ell(f_t(x)) d\mu.$$

For future reference, we note some easy properties of R_μ .

- Proposition 4.1** (properties of R_μ). 1. R_μ is well defined on elements of the universal covering group of $\text{Homeo}_0(M)$.
2. R_μ is a homomorphism when restricted to the subgroup of the universal covering group consisting of paths to μ -preserving homeomorphisms.
3. Identifying $\widetilde{\text{SO}}(2) \cong \mathbb{R}$ with the subgroup of fiber rotations in the universal covering group of $\text{Homeo}_0(M)$. (so that $s \in \mathbb{R}$ is a path through rotations from identity to rotation by $2\pi s$), we have $R_\mu(s) = s$.

Proof. The first assertion follows from the remark that length depends only on homotopy classes rel endpoints. To show the second, suppose that f_t is a path such that f_1 preserves μ . Then for a concatenation of paths $g_t f_t$ we have

$$R_\mu(g_t f_t) := \int_M \ell(g_t \circ f_1(x)) d\mu + \int_M \ell(f_t(x)) d\mu = R(f_t) + R(g_t).$$

which implies the assertion above. The third assertion is immediate from the definition. \square

Proving Theorem 1.3 and Corollary 1.4 Applying this to our situation, let M be a Seifert fibered 3-manifold, let $G = \text{Homeo}_0(M)$ and let \tilde{G} be the covering group corresponding to $\pi_1(G)/\iota(\mathbb{Z})$ as in the proof of Theorem 1.2. Recall that this is *not* necessarily the universal covering group of G , despite our notation \tilde{G} . Let μ be a probability measure on M and let G_μ denote the group of measure preserving homeomorphisms. Let $e \in H^2(G^\delta; \mathbb{Z})$ be a class with nonzero image in $H^2(\text{SO}(2); \mathbb{R})$. By the framework given in Section 2, in order to show that the pullback of e to $H^2(G_\mu^\delta; \mathbb{Z})$ is zero, it suffices to show that $\rho^*(e) = 0$ for any orientable surface Σ and $\rho : \pi_1(\Sigma) \rightarrow G_\mu$.

Let $\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^g [a_i, b_i] \rangle$ and suppose that $\rho : \pi_1(\Sigma) \rightarrow G$ has image in G_μ . Let \tilde{a}_i , and \tilde{b}_i be paths in G from the identity to a_i and b_i respectively. Now we have

$$R_\mu\left(\prod_{i=1}^g [\tilde{a}_i, \tilde{b}_i]\right) = \prod_{i=1}^g [R_\mu(\tilde{a}_i), R_\mu(\tilde{b}_i)] = 0$$

since R_μ is a homomorphism by Proposition 4.1. Now consider the equivalence class of this path $\prod_{i=1}^g [\tilde{a}_i, \tilde{b}_i]$ as an element of \tilde{G} . Since it is a lift of the identity, the definition of \tilde{G} implies that $\prod_{i=1}^g [R_\mu(\tilde{a}_i), R_\mu(\tilde{b}_i)]$ is equivalent to a path obtained by rotating fibers of M by an integral amount, where $n = \rho^*(e) \in \mathbb{Z}$. Since $R(n) = n[F]$ by Proposition 4.1, it follows that $n = 0$, as claimed. \square

To prove Corollary 1.4 it suffices to observe that $\pi_1(T^2)$ is amenable, so any action on a compact manifold M has an invariant probability measure. Thus, any $\rho : \pi_1(T^2) \rightarrow \text{Homeo}_0(M)$ factors through $\rho : \pi_1(T^2) \rightarrow \text{Homeo}_\mu(M) \rightarrow \text{Homeo}_0(M)$ for some probability measure μ , hence $\rho^*(e) = 0$. \square

References

- [1] R. Anderson, *The algebraic simplicity of certain groups of homeomorphisms*. Amer. J. Math. 80 (1958), 955-963.
- [2] R. Bing, *An alternative proof that 3-manifolds can be triangulated*. Ann. of Math. 69 (1959) 37-65.
- [3] M. Bucher, T. Gelande, *Milnor–Wood inequalities for manifolds locally isometric to a product of hyperbolic planes*. Comptes Rendus Math. 346, no. 11-12 (2008), 661-666.
- [4] D. Calegari, *Circular groups, planar groups and the Euler class*. Geometry & Topology Monographs 7 (2004) 431–491.
- [5] A. V. Cernavskii, *Local contractibility of the group of homeomorphisms of a manifold*, (Russian) Mat. Sb. (N.S.) 79.121 (1969), 307–356.
- [6] J.-L. Clerc, B. Ørsted, *The Gromov norm of the Kaehler class and the Maslov index*. Asian J. Math. 7 no. 2 (2003), 269–295
- [7] A. Domic, D. Toledo, *The Gromov norm of the Kaehler class of symmetric domains*. Math. Ann. 276 no. 3 (1987), 425–432.
- [8] R. Edwards, R. Kirby *Deformations of spaces of imbeddings*. Ann. Math. (2) 93 (1971), 63–88.
- [9] M. Gromov. *Volume and bounded cohomology*. Publ. Math. IHES No. 56 (1982), 5–99.
- [10] A. Hatcher, *Homeomorphisms of sufficiently large P^2 -irreducible 3-manifolds* Topology 15 (1976), 343-347
- [11] E. Moise, *Affine structures in 3-manifolds V*. Ann. of Math. 56 (1952) 96-114.
- [12] N. Monod, *An invitation to bounded cohomology*, in International Congress of Mathematicians, vol. II, Eur. Math. Soc., Zurich (2006), 1183–1211.
- [13] S. Hong, J. Kalliongis, D. McCullough and J. H. Rubinstein, *Diffeomorphisms of elliptic 3-manifolds*. Lecture Notes in Math. 2055, Springer, Heidelberg, 2012.
- [14] N. Ivanov, *Diffeomorphism groups of Waldhausen manifolds*. J. Soviet Math. 12, No. 1 (1979), 115-118.
- [15] N. Ivanov. *Homotopies of automorphism spaces of some three-dimensional manifolds*. Dokl. Akad. Nauk SSSR, 244, no. 2, (1979) 274–277.
- [16] K. Mann, C. Rosendal, *The large-scale geometry of homeomorphism groups*. To appear in Ergodic Theory and Dynamical Systems. Available at DOI:<https://doi.org/10.1017/etds.2017.8> (2017)

- [17] D. McCullough, T. Soma. *The Smale conjecture for Seifert fibered spaces with hyperbolic base orbifold*. Journal of Differential Geometry 93.2 (2013), 327-353.
- [18] J. Milnor, *On the existence of a connection with curvature zero*. Comment. Math. Helv. 32 no. 1 (1958), 215-223.
- [19] W. Thurston, *Foliations and groups of diffeomorphisms*. Bull. Amer. Math. Soc. 80 (1974), 304–307.
- [20] J. Wood, *Bundles with totally disconnected structure group*. Comm. Math. Helv. 51 (1971), 183-199.

Kathryn Mann
Dept. Mathematics
Brown University
151 Thayer St. Providence, RI, 02912
kathryn_mann@brown.edu