# 30-minute Test 2

Math 113

November 17, 2015

Name: Solutions

**Instructions:** This test has 4 questions for a total of 24 points. Please show your work in the space provided.

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 7 | |
| 3 | 7 | |
| 4 | 5 | |
| Total: | 24 | |

1. (a) (2 points) Let $R$ be a commutative ring and $I \subset R$ an ideal.
   $I$ is *principal* if I is generated by one. elmt
   $I$ is *prime* means that whenever $ab \in I$, then either $a \in I$ or $b \in I$.

   (b) (3 points) Let $R$ be the ring of all continuous functions from $[0,1]$ to $\mathbb{R}$. Let $I$ be the ideal generated by $f(x) = \sin(x)$ and $g(x) = 3\sin(x)$. Is $I$ principal? Explain why or why not.

   Yes, $3\sin(x) \in (\sin(x))$, so

   $(3\sin(x), \sin(x)) = (\sin(x))$

2. (7 points) Which of the following subsets of $\mathbb{Z}[x]$ are ideals? Circle all the ideals. (you do not need to show any work)

   a) The set of polynomials with constant term 0 (this is $(x)$)

   b) The set of polynomials whose coefficients are multiples of 5 (this is $(5)$)

   c) The kernel of the ring homomorphism $\phi : \mathbb{Z}[x] \to \mathbb{Z}$ given by evaluating at 2 (i.e. $\phi(p(x)) = p(2)$).
   Any kernel is an ideal

   d) $\{p(x) \in \mathbb{Z}[x] : p(x) \text{ has degree } \leq 3\}$
   Not even a subring!

   e) $\{0\}$
   Always an ideal

3. (a) (1 point) Let $R$ be a ring with 1. What does it mean for $r \in R$ to be a *unit*?

There is $a \in R$ such that $ar = 1$

(b) (2 points) If $r$ is a unit in a ring $R$, is $\left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right)$ a unit in $M_2(R)$?
Explain briefly why or why not.

No, $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ 0 & 0 \end{pmatrix}$

since $0 \neq 1$ this can't be the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) (4 points) Let $R$ be a commutative ring with 1, and $a, b \in \mathbb{R}$. Assume that $(a) = (b)$ and $a$ is not a zero divisor. Prove that $a = bu$ for some unit $u$.

Suppose $(a) = (b)$.
Then $a \in (b)$ so $a = bc$ for some $c \in R$
Also $b \in (a)$ so $b = ad$ for some $d$

Substituting, we have $a = adc$

Since $a$ is not a zero divisor, we can use cancellation and
$$1 = dc.$$

Thus, $c$ is a unit and $a = bc$.

4. (a) (2 points) Let $f(x) = 2x$ and $g(x) = 12x^2 - 4x + 2$ in the ring $\mathbb{Q}[x]$.
Write $g(x) = q(x)f(x) + r(x)$ where the norm (i.e. degree) of $r(x)$ is less than that of $q(x)$

$$12x^2 - 4x + 2 = (6x - 2)(2x) + 2$$

(b) (1 point) Is the polynomial $r(x)$ that you found above a greatest common divisor (G.C.D.) of $f(x)$ and $g(x)$? (yes or no)

in $\mathbb{Q}[x]$

YES!

(c) (1 point) If your answer above was "no", find a G.C.D. If it was "yes", explain why you said yes.

It divides both $f$ and $g$, and no polynomial of higher degree divides both $f$ and $g$.

(d) (1 point) Is 3 a G.C.D. of $f(x)$ and $g(x)$ in $\mathbb{Q}[x]$? (yes or no)

Yes, (3 is a unit in $\mathbb{Q}[x]$).