SOLUTIONS.

Math 113 Test 1

1. (3 points) Definition (fill in the blanks):

Let G be a group and A a set. An *action of G on A* is a map $G \times A \to A$,
satisfying two conditions:

i) $(g_1 g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$ for all $g_1, g_2 \in G$, $a \in A$

ii) $e \cdot a = a$ for all $a \in A$

2. (7 points) Which of the following sets are subgroups the group $(\mathbb{R} - \{0\}, \times)$?
Circle all of the examples that are subgroups. (you do not need to show work)

a) The set of even integers doesn't contain 1.

b) The set of positive rational numbers Contains 1, closed under $\times$ and inverse

c) $\{g \in \mathbb{R} - \{0\} \mid g = g^{-1}\}$ $= \{1, -1\}$

d) The set of all integer powers of 3. contains $1 = 3^0$
inverse of 3^n is 3^{-n}

e) $\{0\}$ not in $\mathbb{R} - \{0\}$

3. Let G be a [illegible] group and $g \in G$. Define $\phi : \mathbb{Z} \to G$ by $\phi(n) = g^{2n}$

(a) (3 points) Prove that ϕ is a homomorphism.

$$\phi(n+m) = g^{2(n+m)} = g^{2n} g^{2m} = \phi(n)\,\phi(m)$$

(b) (3 points) Prove that the image of ϕ is a cyclic subgroup

If h is in the image of ϕ,

then $h = g^{2n}$ for some $n \in \mathbb{Z}$

$= (g^2)^n$ for some $n \in \mathbb{Z}$.

Conversely, each element of $\phi(\mathbb{Z})$ is of the form $(g^2)^n$

This shows $\phi(\mathbb{Z})$ is generated by g^2, and is a subgroup since $(g^{2n})(g^{2m})^{-1} = g^{2(m-n)}$.

(c) (1 point) Define the *order* of a group.

$|G|$ is the number of elements in G, if finite and infinite otherwise.

(d) (2 points) If $|G| = 10$, is it necessarily true that the order of the image of ϕ is 10? ~~Explain in one sentence~~. Prove or give a counterexample.

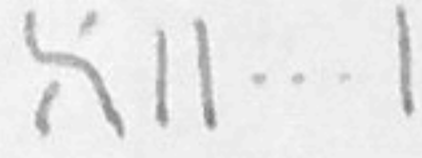
No, consider the case $G = \mathbb{Z}/10\mathbb{Z}$.

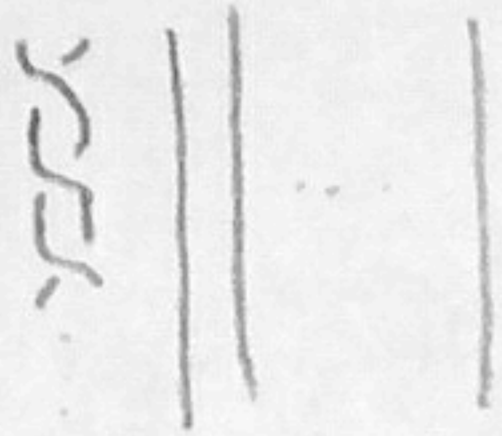
Then $\phi(\mathbb{Z}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$.

(other counterexamples are possible)

4. (4 points) Show that the braid group B_n is not isomorphic to the ~~symmetric~~ dihedral group D_{2n}. If you use any facts we proved in class or that are from the book, state these facts clearly.

For each element $g \in D_{2n}$, $|g|$ is finite, since D_{2n} is finite.

However, consider $b_1 =$  $\in B_n$.

$(b_1)^n =$  $\neq id$, so b_1 has infinite order.

Since $|g| = |\phi(g)|$ for any isomorphism ϕ, there can be no such isomorphism $B_n \to D_{2n}$. $\phi(b_1)$ would have to have infinite order.