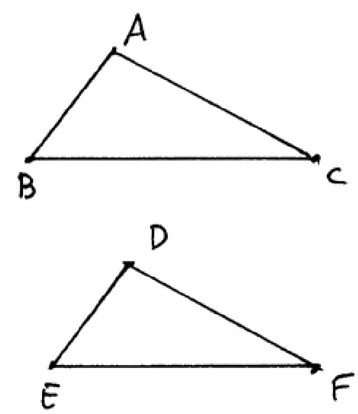
we will have to include some axiom that guarantees the existence of the intersection points of circles with other circles, or with lines, at least those that arise in the ruler and compass constructions of Euclid's *Elements*. Some modern axiom systems (such as Birkhoff (1932) or the School Mathematics Study Group geometry) build the real numbers into the axioms with a postulate of line measure, or include Dedekind's axiom that essentially guarantees that we are working over the real numbers. In this book, however, we will reject such axioms as not being in the spirit of classical geometry, and we will introduce only those purely geometric axioms that are needed to lay a rigorous foundation for Euclid's *Elements*.

The issue of intersecting circles arises again in (I.22), where Euclid wishes to construct a triangle whose sides should be equal to three given line segments a, b, c. This requires that a circle with radius a at one endpoint of the segment b should meet a circle of radius c at the other end of the segment b. Euclid correctly puts the necessary and sufficient condition that this intersection should exist in the statement of the proposition, namely that any two of the line segments should be greater than the third. However, he never alludes to this hypothesis in his proof, so that we do not see in what way this hypothesis implies the existence of the intersection point. While some commentators have criticized Euclid for this, Simson ridicules them, saying "For who is so dull, though only beginning to learn the Elements, as not to perceive ... that these circles must meet one another because FD and GH are together greater than FG." Still, Simson has only discussed the position of the circles and has not addressed the second issue of why the intersection point exists. (See Plate V, p. 109)

The Method of Superposition



Let us look at the proof of (I.4), the side–angle–side criterion for congruence of two triangles (SAS for short). Suppose that $AB = DE$, and $AC = DF$, and the included angle $\angle BAC$ equals $\angle EDF$. We wish to conclude that the triangles are congruent, that is to say, the remaining sides and pairs of angles are congruent to each other, respectively. Euclid's method is to "apply the triangle" ABC to the triangle DEF. That is, he imagines moving the triangle ABC onto the triangle DEF, so that the point A lands on the point D, and the side AB lands on the side DE. Then he goes on to argue that the ray AC must land on the ray DF, because the angles are equal, and hence C must land on F because the sides are equal. From here he concludes that the triangles coincide entirely, hence are congruent.

This is another situation where Euclid is using a method that is not explicitly allowed by his axioms. Nothing in the Postulates or Common Notions says that we may pick up a figure and move it to another position. We call this the *method of superposition*.

Euclid uses this method again in the proof of (I.8), but it appears that he was reluctant to use it more widely, because it does not appear elsewhere. If it were a generally accepted method, for example, then Postulate 4, that all right angles are equal to each other, would be unnecessary, because that would follow easily from superposition.

If we think about the implications of this method, it has far-reaching consequences. It implies that one can move figures from one part of the plane to another without changing their sides or angles. Thus it implies a certain homogeneity of the geometry: The local behavior of figures in one part of the plane is the same as in another part of the plane. If you think of modern theories of cosmology, where the curvature of space changes depending on the presence of large gravitational masses, this is a nontrivial assumption about our geometry.

To state more precisely what assumptions the method of superposition is based on, let us define a *rigid motion* of the plane to be a one-to-one transformation of the points of the plane to itself that preserves straight lines and such that segments and angles are carried into congruent segments and angles. To carry out the method of superposition, we need to assume that there exist sufficiently many rigid motions of our plane that

(a) we can take any point to any other point,
(b) we can rotate around any given point, so that one ray at that point is taken to any other ray at that point, and
(c) we can reflect in any line so as to interchange points on opposite sides of the line.

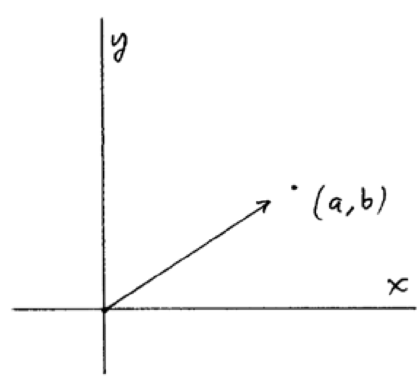
If we were working in the real Cartesian plane $\mathbb{R}^2$ with coordinates x, y, we could easily show the existence of sufficient rigid motions by using *translations*, *rotations*, and *reflections* defined by suitable formulas in the coordinates.

For example, a translation taking the point $(0,0)$ to (a,b) is given by

$$\begin{cases} x' = x + a, \\ y' = y + b, \end{cases}$$

and a rotation of angle α around the origin is given by

$$\begin{cases} x' = x\cos\alpha - y\sin\alpha, \\ y' = x\sin\alpha + y\cos\alpha. \end{cases}$$



Thus we can easily justify the use of the method of superposition in the real Cartesian plane. However, since there are no coordinates and no real numbers in Euclid's geometry, we must regard his use of the method of superposition as an additional unstated postulate or axiom.

To formalize this, we could postulate the existence of a group of rigid motions acting on the plane and satisfying the conditions (a), (b), (c) mentioned above. Indeed, there is an extensive modern school of thought, exemplified by Felix Klein's *Erlanger Programm* in the late nineteenth century, which bases the study of geometry on the groups of transformations that are allowed to act on the geometry. This point of view has had wide-ranging applications in differential geometry and in the theory of relativity, for example.

We will discuss the rigid motions in Euclidean geometry in greater detail later (Section 17). For the moment let us just note that the proof of the (SAS) criterion for congruence in (I.4) requires something more than what is in Euclid's axiom system. Hilbert's axioms for geometry actually take (SAS) as an axiom in itself. This seems more in keeping with the elementary nature of Euclid's geometry than postulating the existence of a large group of rigid motions.

Finally let us note that Euclid's use of the method of superposition in the proof of (I.4) gives us some more insight into his concepts of "equality" for line segments and angles. In Common Notion 4 he says that things that coincide with one another are equal (congruent) to one another. In the proof of (I.4) he also uses the converse, namely, if things (line segments or angles) are equal to one another (congruent), then they will coincide when one is moved so as to be superimposed on the other. So it appears that Euclid thought of line segments or angles being congruent if and only if they could be moved in position so as to coincide with each other.

Betweenness

Questions of betweenness, when one point is between two others on a line, or when a line through a point lies inside an angle at that point, play an important, if unarticulated, role in Euclid's *Elements*. To explain the notion of points on a line lying between each other, one could simply postulate the existence of a linear ordering of the points. Similarly, for angles at a point one could talk of a circular ordering.

But when a hypothesis of relative position of points and lines in one part of a diagram implies a relationship for other parts of the figure far away, it seems clear that something important is happening, and it may be dangerous to rely on intuition.

For example, how do you know that the angle bisector at a vertex A of a triangle ABC meets the opposite side BC between the points BC and not outside? Of course, it is obvious from the picture, but what if you had to explain why without drawing a picture?