What is geometry?

a walk through mathematical spaces

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Sonoma State M*A*T*H Colloquium
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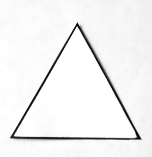
What do modern geometers study?

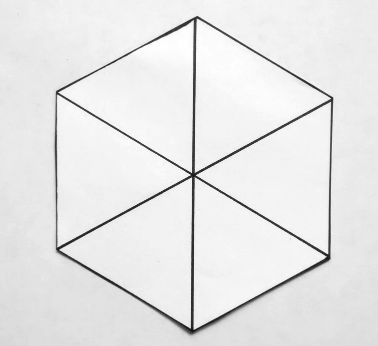
What *spaces* appear in mathematics?

How do we find them?

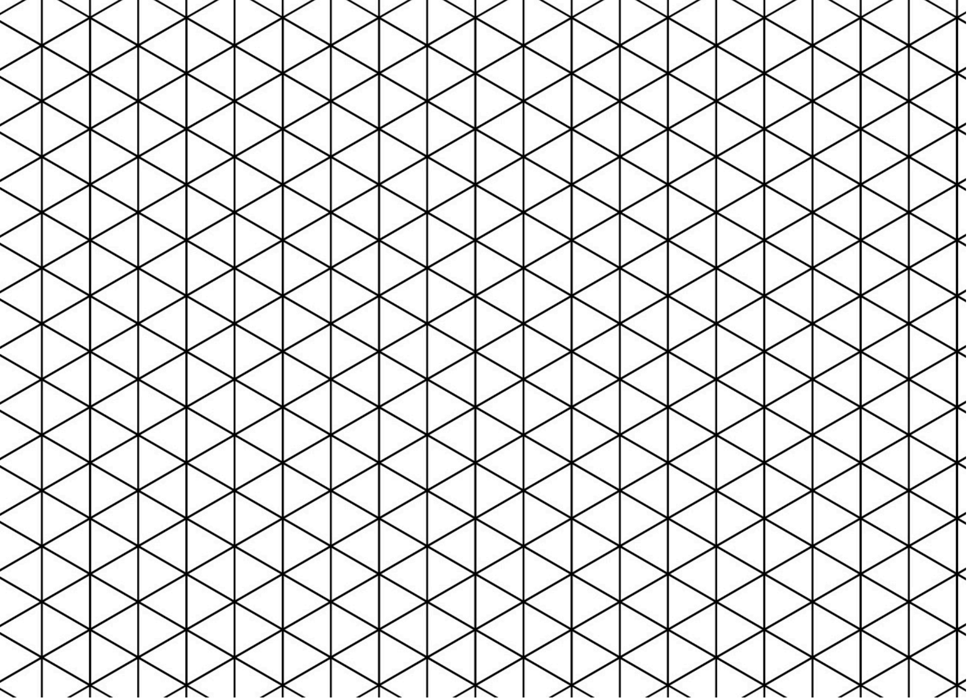
1. Curiosity

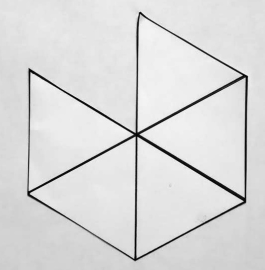
Playing with paper





6 around a vertex

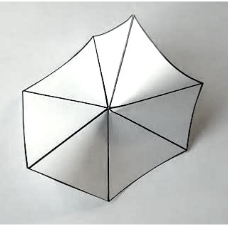


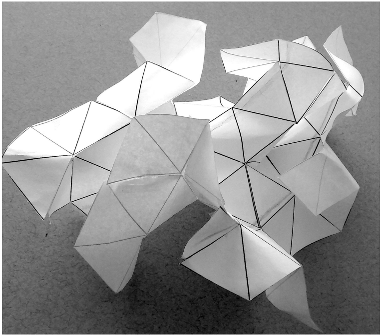


5 around a vertex?

icosahedron

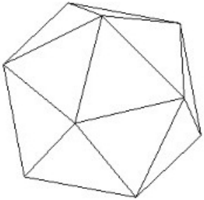
7 around a vertex (?!)



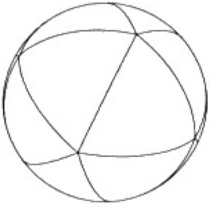


hyperbolic plane

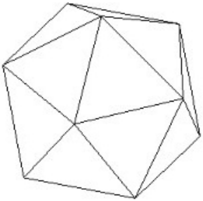
Smooth versions



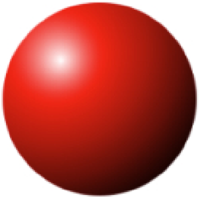
$\longrightarrow$



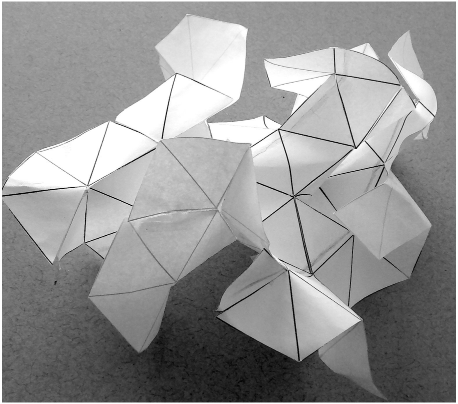
Smooth versions



$\longrightarrow$



Smooth versions



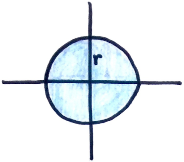
$\longrightarrow$

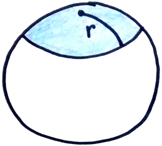
Smooth versions

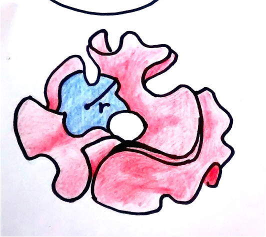
Smooth versions

Let's ~~explore~~ do geometry

Area of disc of radius r







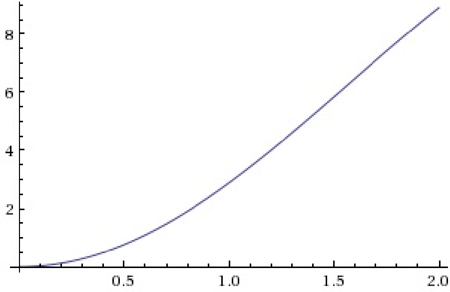
Plane: πr^2 quadratic

Sphere: $2\pi(1 - \cos(r)) \simeq$ linear

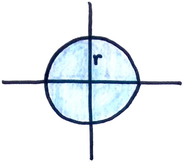
.

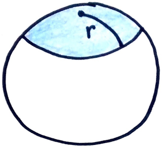
plot	$2\,\pi\,(1 - \cos(r))$	$r = 0$ to 2

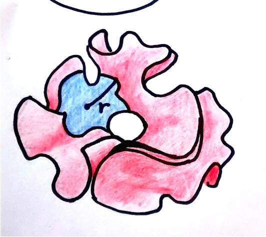
Plot:



Area of disc of radius r







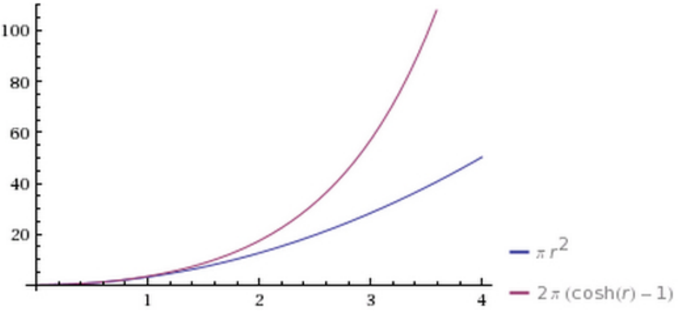
Plane: πr^2 quadratic

Sphere: $2\pi(1 - \cos(r)) \simeq$ linear

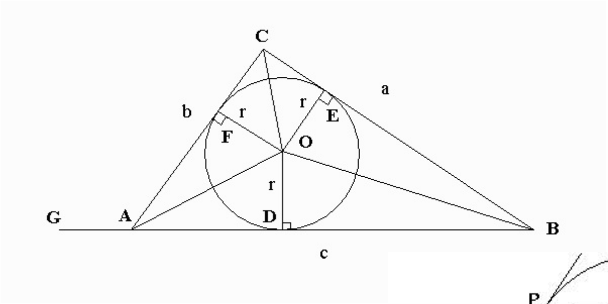
Hyperbolic space: $2\pi(\cosh(r) - 1) \simeq e^r$
exponential

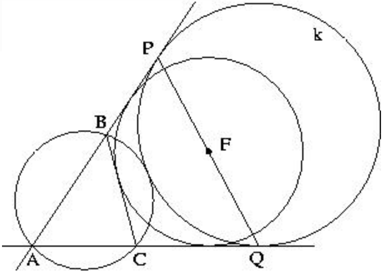
"there's lot's of space in hyperbolic space"

Plot:



Next goal: high school geometry





Straight lines in curved spaces

Definition

A *geodesic* is a "shortest path" between two points.

Geodesics play the role of straight lines.

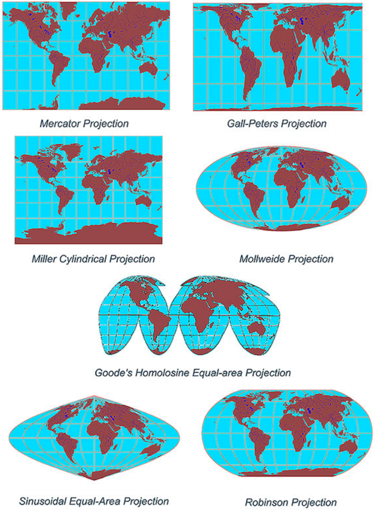
Plane – straight lines

Sphere – segments of great circles

Hyperbolic space – ??

A better picture

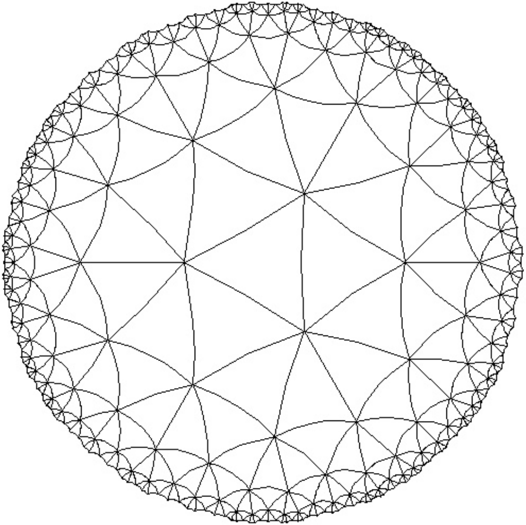
We have many ways of representing the (spherical) earth as a flat picture



all of them are distorted

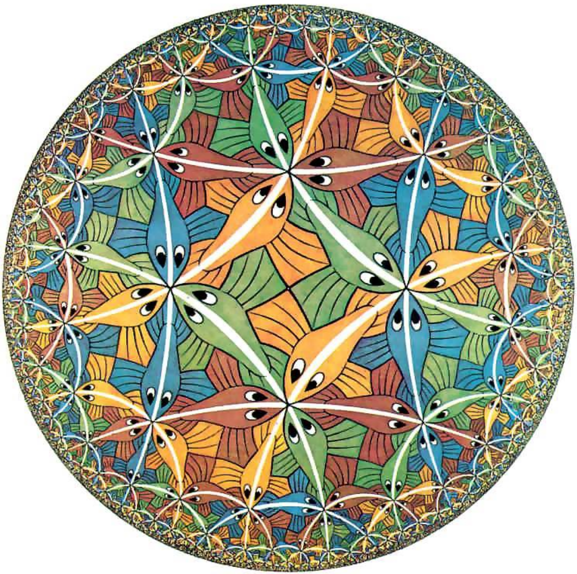
A better picture

The *Poincaré disc model* does the same for hyperbolic space.

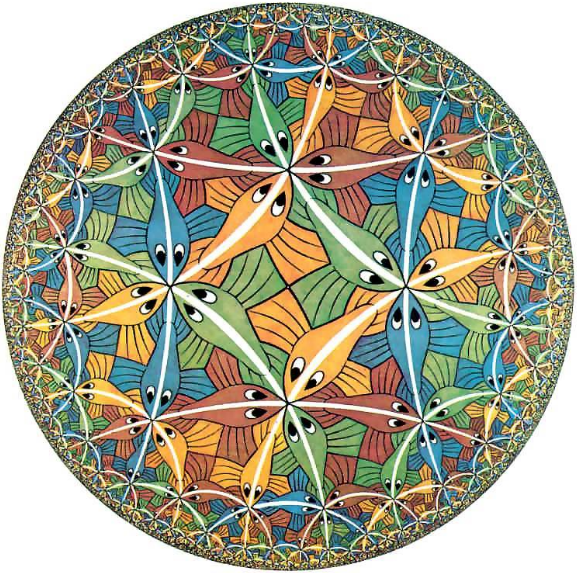


(7 triangles around a vertex)

Familiar?



Geodesics



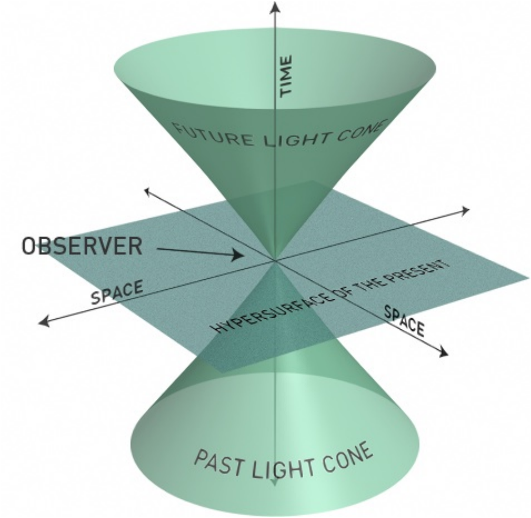
Trigonometry and more...

sine law in hyperbolic space: $sin(a) = \frac{sinh(\text{opposite})}{sinh(\text{hypotenuse})}$

trigonometry, angle sum of a triangle $= \pi$-area, two-column proofs, etc....

2. Discovery

The (Minkowski–Einstein) space-time universe



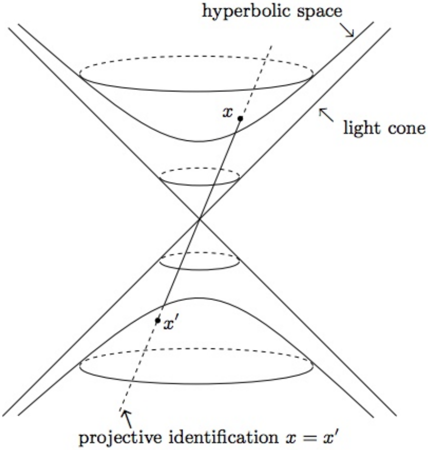


Figure 1. Minkowski space.

from Cannon–Floyd–Kenyon–Parry *Hyperbolic space* [**?**].

3. Abstraction

Geometry today

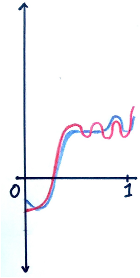
Metric space = collection of objects
+ notion of "distance" between them

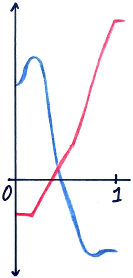
Geometry today

Metric space = any collection of objects
+ notion of "distance" between them

Example 1: Objects = all continuous functions $[0, 1] \to \mathbb{R}$

Distance?





$$\mathrm{dist}(f, g) = \sup_{x \in [0,1]} \{|f(x) - g(x)|\}$$

Geometry today

Metric space = any collection of objects
+ notion of "distance" between them

Example 1: Objects = all continuous functions $[0, 1] \to \mathbb{R}$

Direct application: Solution of ODE's

$$y'(t) = F(y(t)),\ y(0) = a$$

Picard's theorem: iterative process to get closer and closer to solution $y(t)$.

Works for functions to $\mathbb{R}^n$ too!

Example 2:

Objects $= n \times n$ matrices

Distance: $d(A, B) = \max \|a_{ij} - b_{ij}\|$

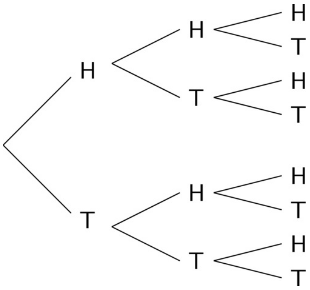
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ close to } \begin{pmatrix} 1.2 & 0 & 0.03 \\ 0 & 1 & 0.1 \\ 0 & 0.05 & 0.99 \end{pmatrix}$$

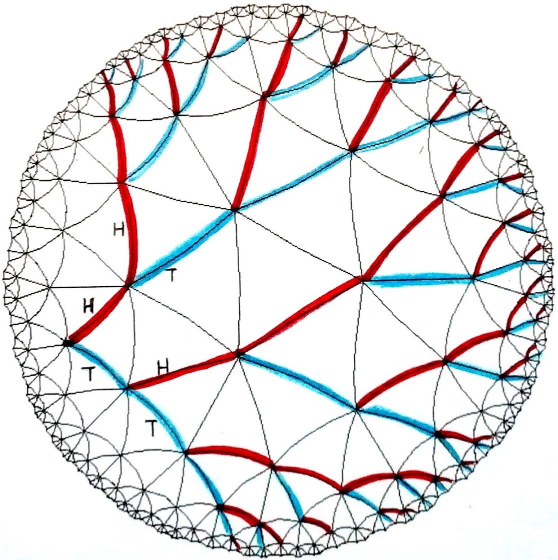
Remark: Not the only nice way to make a distance!

Riemannian geometry provides a framework to define metrics that "remember" matrix multiplication:

$$d(A, B) = d(CA, CB)$$

Example 3: space of all outcomes

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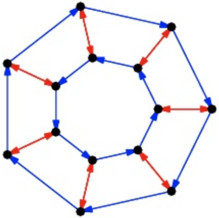
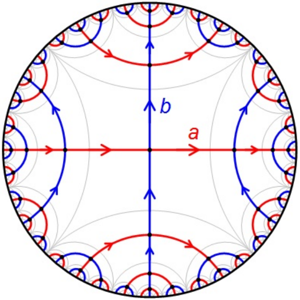
Advanced Example: Objects = Elements of a group

(with specified generators, relations)



$D_{14} \quad \langle r, s \mid s^2 = 1, d^7 = 1, rs = sr^1 \rangle$

space/distances depicted by *graph*



$F_2 \quad \langle a, b \rangle$

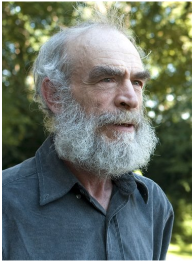
4. So what's my research?

One answer: *geometric group theory*

Geometric group theory

The study of algebraic objects as geometric objects

"groups"



M. Gromov

1987 paper *"Hyperbolic groups"*

Philosophy/perspective has now been used in:

low-dimensional topology, manifold theory, algebraic topology, complex dynamics, combinatorial group theory, algebra, logic, classical families of groups, Riemannian geometry, representation theory... ...and my work too!

References and further reading

[1] D. Henderson, S. Taimioa, *Crocheting the Hyperbolic Plane*. Mathematical Intelligencer, Vol. 23, No 2 (2001) 17–28

[2] J. Cannon, W. Floyd, R. Kenyon, W. R. Parry, *Hyperbolic Geometry*. in *Flavors of Geometry*, MSRI Publications Volume 31, 1997 29–115

[3] W. Thurston, *Three-Dimensional Geometry and Topology*. Princeton university press, 1997.
(a wonderful introduction to hyperbolic geometry and much more)

[4] J. Weeks, *The shape of space*. CRC Press, 2001.
(also highly recommended and a little easier than Thurston)

[5] Videos:
"The shape of space" inspired this video:
`http://www.geom.uiuc.edu/video/sos/`

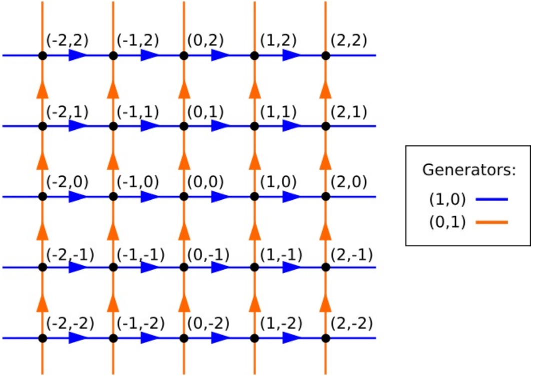
More hyperbolic geometry in *"not knot"*
`http://www.geom.uiuc.edu/video/NotKnot/`

"Dimensions" a newer series of videos with a geometric perspective:
`http://www.dimensions-math.org/`

[6] JUST FOR FUN: *"Do–It–Yourself Hyperbolic Geometry"* class notes on my webpage: `https://math.berkeley.edu/~kpmann/DIYhyp.pdf`

Geometric group theory gallery

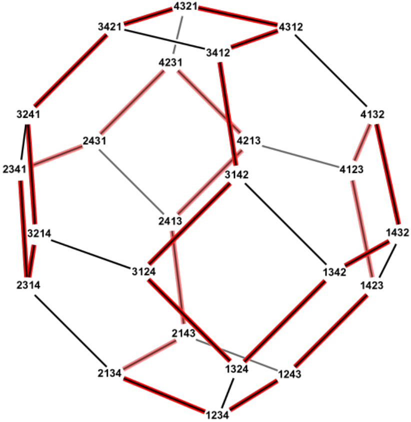
The group $\mathbb{Z} \times \mathbb{Z}$



from `math.cornell.edu/~mec/2008-2009/Victor/part4.htm`

Geometric group theory gallery

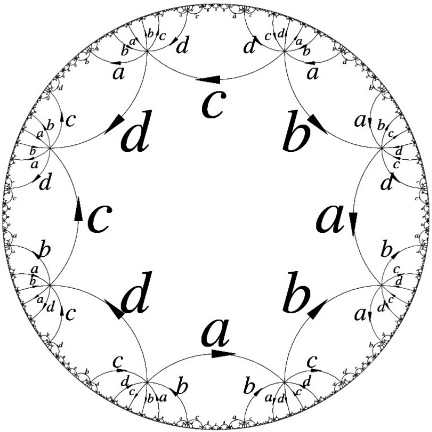
The symmetric group S^4 (generated by transpositions)



wikimedia commons

Geometric group theory gallery

The group $\langle a, b, c, d, | aba^{-1}b^{-1} = c^{-1}d^{-1}cd \rangle$



from `yann-ollivier.org/maths/primer.php`

Geometric group theory gallery

A group that looks like *3-dimensional* hyperbolic space (!)

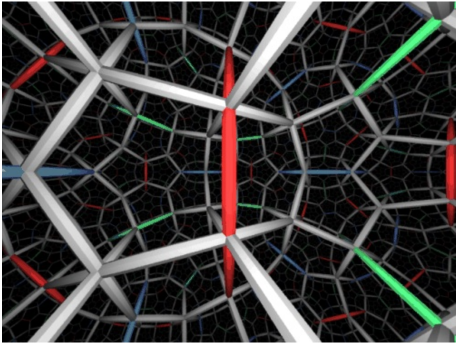


image from the cover of [**?**]

Thanks!

and please come play with my hyperbolic spaces