Ham sandwich Theorem Proof outline:

Let A_1, A_2, A_3 be solids in $\mathbb{R}^3$. We'll find a plane P that bisects all of them.

Following the hint, define $g_1 : S^2 \to \mathbb{R}$ by $g_1(x) = v_1(x) - w_1(x)$, where $v_1(x)$ is the volume of A_1 on the side of plane P_x that contains x, and $w_1(x)$ the volume of A_1 on the other side — but we still need to specify which plane P_x is (there are many planes that have normal vector x). So let P_x be a plane such that P_x has normal vector x, and P_x bisects A_3.
(Such a plane exists by the intermediate value theorem).
Assuming A_3 is nice (e.g. smooth boundary, can be approximated by rectilinear figures) P_x will vary smoothly with x (I did not expect you to prove this rigorously). So g_1 is smooth also.
Similarly, let $g_2(x) = v_2(x) - w_2(x)$, where $v_2(x)$ is the volume of A_2 on the $\vec{x}$ side of P_x, and $w_2(x)$ the volume of A_2 on the other side. Note $g_i(-x) = -g_i(x)$
By Borsuk-Ulam consequence # 2, the functions g_1 and g_2 have a common zero. For this x, P_x bisects all three solids.

Since $P_{-x} = P_x$, and $v_i(x) = w_i(-x)$

so $g_i(-x) = v_i(-x) - w_i(-x)$

$= w_i(x) - v_i(x) = -g_i(x)$