Exercises on left and circularly ordered groups

1. Let G be the fundamental group of the Klein bottle, it has presentation
$$G = \langle a, b \mid aba^{-1} = b^{-1} \rangle$$
Show that there are exactly 4 different left orders on G.

2. Describe all left orders on $\mathbb{Z} \times \mathbb{Z}$. (hint: there are infinitely many, but they admit a nice description!)

3. As a follow up to the previous question, show that for any finite set $S \subset \mathbb{Z} \times \mathbb{Z}$, there are at least two different left orders on $\mathbb{Z} \times \mathbb{Z}$ that agree on all elements of S.

4. Suppose G is generated by a and b. Find examples of relators that prevent G from being left orderable. (we know $a^n = 1$ is such an example, but you can avoid torsion!)

5. Describe all circular orders on $\mathbb{Z}$.

6. Show the mapping class group of a surface with marked point has many circular orders

7. Can you do problem 4 for circular orders?

8. Let $1 \to K \to G \to H \to 1$ be a short exact sequence, and assume K and H are left ordered. Use this to produce a left order on G.
Now assume K is left ordered and H circularly ordered. Use this to produce a circular order on G.

9. (challenge) Suppose A and B are left (or circularly) orderable. Give conditions on the subgroup C that ensure $A *_C B$ is left (respectively, circularly) orderable.

10. Show that $\text{Homeo}_+(S^1)$, the group of orientation preserving homeomorphisms of the circle, has a circular order.

References for further reading:

1. *Right-Ordered Groups* by V. Kopytov and N. Medvedev
2. *Groups, Orders, Dynamics* by B. Deroin, A. Navas, and C. Rivas
3. *Circular groups, planar groups, and the Euler class* by D. Calegari
4. *Groups acting on the circle* by E. Ghys