Reading list on cohomology of diffeomorphism groups and related topics

1 Introductory talks

1. $H_1(\mathrm{Diff}_c(M)) = 0$.
    - Haller, Rybicki, Teichmann. *Smooth perfectness for the group of diffeomorphisms* (2012). See also the expository simplified version (Mann, 2012).
    - Banyaga, *The structure of classical diffeomorphism groups* (book). (Chapter 2 gives some of the history of the problem, and a proof following some of Thurston's arguments).
2. Segal's proof of Mather's theorem.
    - Segal *Classifying spaces related to foliations.* (1978)

2 Introduction to Godbillon-Vey Classes

Definition. Thurstons example that G-V is a continuous class (explained very well in Moritas book). Botts examples. References:

- Morita: Geometry of characteristic classes (book).
- Bott: Lectures on characteristic classes and foliations
- Harsh V Pittie. Characteristic classes of foliations.

3 Gelfand-Fuks cohomology

We'll spend a couple of lectures surveying the work of Bott and Morita.

1. Bott's work (1975-79)
    - *Notes on Gelfand-Fuks cohomology and characteristic classes*
      Very readable. Bott proves that GF cohomology is finite dimensional in each degree and he formulates a conjecture that GF cohomology is the same as a cohomology of some section space.
    - *Some formulas for the characteristic classes of group actions*
      Bott proves Thurston's formula for characteristic class of S^1 bundles
    - (with Segal) *On the cohomology of the vector fields of manifolds*
      A very important paper, in which they give a proof of Botts conjecture.
    - *On the characteristic classes of groups of diffeomorphisms*
2. Morita
    - *Nontriviality of the Gel'fand Fuchs characteristic classes for flat S^1 bundles* (1984).
      Morita uses minimal model to prove that powers of Euler class dont vanish in the cohomology of $\mathrm{Diff}^\infty(S^1)$.

4 Homology and cohomology of diffeomorphism groups and groups of symplectomorphisms. Characteristic classes of flat bundles.

Topics covered here will depend on interest of the group. We'll definitely have a lecture on Ghys-Sergiescu as a simple example of the McDuff-Segal method. It would be nice to have an interested person present Dupont's first paper.

1. McDuff's work
   - *The homology of some groups of diffeomorphisms* (1980)
     This is a nice paper using important techniques of Segal, and a categorial approach. Interestingly, the Ghys-Sergiescu paper (below) uses the same techniques.
   - *Symplectic diffeomorphisms and the flux homomorphism* (1984)
     In this paper, McDuff proves that the group of symplectic diffeomrphisms is not homotopy equivalent to $\mathrm{Diff}(M)$ for certain class of open symplectic manifolds.
2. Dupont's work on characteristic classes of flat bundles
   - *Simplicial de Rham cohomology and characteristic classes of flat bundles* (1976)
     (very readable. Develops de Rham theory of simplicial manifolds and an integral formula for characteritic class of flat bundles, as applications he gives simple proofs of Bott vanishing and the Milnor-Wood inequality)
   - *Characteristic classes for flat bundles and their formulas. (1994)*
3. Ghys-Sergiescu
   - *Sur un groupe remarquable de diffeomorphismes du cercle* (1987)
     Calculates the cohomology of Thompsons group inside of $\mathrm{Homeo}(S^1)$. Shows that it is generated by the Euler class and a Godbillon vey class. Related to McDuff-Segal above.
4. Optional additional topics:
   - Milnor, *On the homology of Lie groups made discrete.*
     Not on diffeomorphism groups, but might be a good intro to homology of discrete groups.
   - Bowden *Flat structures on surface bundles* (2011).
     Relates the Euler class to the Calabi homomorphism. Computes H_1 of the diffeomorphism group of a surface with marked points.
   - Morita's work on discontinuous invariants of diffeomorphism groups. Discontinuous invariant is a technique devised by Morita to use the uncountability of homology with integral coefficients to construct even more nontrivial cohomology classes.
   - Morita and Kotschick's more recent work on characteristic classes of foliated surface bundles.
     References: *Signatures of foliated surface bundles and the symplectomorphism group of surfaces* (2005). (In this paper, they prove that powers of the first MMM-class are nonzero for a flat surface bundle over a surface whose holonomy lands in symplectomorphism group of surfaces).
     Crossed flux homomorphisms and vanishing theorems (2006), *Characteristic classes of foliated surface bundles with area preserving holonomy* (2006).

5 Characteristic classes of foliations (some of this overlaps with the above)

1. Segal's work on the classifying space of foliations.
   We'll hear a bit about this in the introductory talk. McDuff's work very much relied on Segals work, so we'll get some flavor for it talking about McDuff as well.

2. Thurston's work on existence and classification of foliations. (mid 1970s)
   Haefliger proved, for open manifolds, that homotopy classes of maps from M to Haefliger's classifying space $B\Gamma_k$ classify codimension-k foliations on M. Thurston did this for closed manifolds (which is the much harder problem). The $k = 1$ case is a bit different and comes from Mather. The basic statement of the problem is very well framed in Bott's ICM address from 1970. The references for Thurston are:

   - *Foliations and groups of diffeomorphisms* (research announcement)
   - *The theory of foliations of codimension greater than 1* (section 2, the construction, is hard to understand, but there is an easier version of a similar result in the paper called *a local construction for foliations of 3-manifolds*)

3. Hurder's work on characteristic classes of foliations (optional?)

   - dual homotopy invariants of foliations (1981)
   - the secondary classes of foliations with trivial normal bundles. (1981)
   - Homotopy characteristic classes of foliations. (1990)
     Hurder used Sullivans minimal model in these papers to show there are lots independent secondary characteristic classes that vary continuously. Hurder also has a survey paper (2009) called *Classifying foliations* that might be of interest.

6 Realization problems

The main theorem of Morita's "Characteristic classes of surface bundles" (1987) together with Bott vanishing implies that the mapping class group of a surface cannot be realized by diffeomorphisms. There are more recent works that prove this by very different methods:

- Markovic, *Realization of the mapping class group by homeomorphisms*, (2007).
  Markovic shows that the mapping class group cannot be realized as a group of *homeomorphisms*. We do not know a cohomological proof of this! Markovic's paper is quite difficult. An easier to read version has been written by Le Calvez (*A periodicity condition and the section problem on the mapping class group*, available on the arxiv) but even this is still a long and somewhat difficult paper.

- Franks and Handel have a dynamical argument that works for C^1 diffeomorphisms, in *Global fixed points for centralizers and Moritas theorem* (2009)

- In the case of the mapping class group of a surface with a marked point, a cohomological proof is given by Bestvina-Church-Souto in *Some groups of mapping classes not realized by diffeomorphisms* (2009).
  This argument has been generalized greatly by Tshishiku in *Cohomological obstructions to Nielsen realization* (2014).

7 Continuous vs. discrete groups

1. Hurtado (http://arxiv.org/abs/1307.4447) shows, using geometric techniques, that any homomorphism between diffeomorphism groups of manifolds is continuous. This implies that any homomorphism $\mathrm{Diff}(M) \to \mathrm{Diff}(N)$ induces a map on classifying spaces $B\,\mathrm{Diff}(M) \to B\,\mathrm{Diff}(N)$

8 Bounded Cohomology (optional topic)

1. Ghys, *Groupes d'homéomorphismes du cercle et cohomologie bornée.* (1987)
   This paper proves the beautiful result that a discrete group has an element of second bounded cohomology ($\mathbb{Z}$ coefficients) with a cocycle representative taking only the values 0 and 1 if and only if the group acts by homeomorphisms on the circle. In this case, the cocycle is the pullback of the Euler class in $H^2(\mathrm{Homeo}(S^1), \mathbb{Z})$. It's an important result and quite readable.

   Ghys techniques are more dynamical than homological or algebraic. I recommend also looking at the survey paper An invitation to bounded cohomology by N. Monod. (for an idea of what else bounded cohomology is good for). If this seems interesting, Iozzi has a paper on bounded cohomology and homeomorphisms of the circle that uses some sophisticated recent techniques: Bounded cohomology, boundary maps, and rigidity of representations into $\mathrm{Homeo}_+(S^1)$ and $SU(1, n)$