Math 113 Practice Questions for Final Exam.

These practice questions cover Sylow's theorem and "actions" from group theory, as well as rings and fields. For practice questions on group theory, I recommend you re-write the first midterm.

Some of these questions may be more time consuming than the questions on the final exam. If you can solve the problems here (no time limit), that indicates that you will be in good shape to do well on the exam.

1. (a) Define *principal ideal*
   (b) Show that the ideal of $\mathbb{Z}[x]$ generated by 2 and x is not principal.
   (c) Does part a) tell you whether $\mathbb{Z}[x]$ is a Euclidean domain? Explain.

2. Let $p(x) = x^3 + 3x^2 + 2x + 4$. Show that $p(x)$ is reducible in $\mathbb{Z}/7\mathbb{Z}[x]$ but not in $\mathbb{Z}/3\mathbb{Z}[x]$. Can you deduce from this whether $p(x)$ is reducible in $\mathbb{Z}[x]$?

3. (a) Define what it means for an element of a ring to be prime. Is prime necessarily the same thing as irreducible?
   (b) Factor $17i$ into a product of irreducible elements in $\mathbb{Z}[i]$. Are there any other possible ways to factor $17i$ into irreducible elements other than what you wrote down? If so, give an example.

4. Use the Euclidean algorithm for $\mathbb{Q}[x]$ (recall that the norm of a polynomial is its degree) to find a GCD of $x^3 - 2$ and $3x - 1$ in $\mathbb{Q}[x]$

5. Show that $\phi : \mathbb{Z}[\sqrt{5}] \to \mathbb{Z}[\sqrt{5}]$ defined by $\phi(a + b\sqrt{5}) = a - b\sqrt{5}$ is a ring homomorphism. What is the kernel of ϕ?

6. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices. Is the set of diagonal matrices a subring of $M_n(\mathbb{R})$? Is the set of *diagonal matrices* a (left, right or two-sided) ideal?

7. Let $R = \mathbb{Z}/6\mathbb{Z}[x]$ and let I be the ideal generated by x^3. How many elements does R/I have? Is R/I an integral domain?

8. Let $F = \{a + b\sqrt{5} + c\sqrt{2} + d\sqrt{10} \mid a, b, c, d \in \mathbb{Q}\}$
   (a) Show that F is a field.
   (b) What is the degree of the field extension $\mathbb{Q} \subset F$?
   (c) Give an example of a subfield of F that is not $\mathbb{Q}$
   (d) Is $\mathbb{Z}/5\mathbb{Z}$ a subfield of $\mathbb{Q}$?

9. Let G be a group of order 36.
   (a) Prove that G has either 1 or 4 Sylow 3-subgroups.
   (b) If G has only 1 Sylow 3-subgroup, why must this subgroup be normal?
   (c) Now suppose that G has 4 Sylow 3-subgroups, call them H_1, H_2, H_3 and H_4. Let A be the set $\{H_1, H_2, H_3, H_4\}$ Show that
   $$g \cdot H_i = gH_ig^{-1}$$
   defines an *action* of G on A
   (d) (BONUS QUESTION) Recall that the *kernel* of an action is $\{g \in G \mid ga = a \text{ for all } a \in A\}$ and is a subgroup of G. Show that the kernel of the action defined in part (c) is a proper subgroup of G that is not $\{e\}$. Deduce that G has a nontrivial normal subgroup that is not $\{e\}$.