Math 141 Practice Exam SOLUTIONS

Notes: This practice exam is a version of the Fall 2014 Math 141 final exam, modified to match the material that we covered in our class. It should be roughly comparable in length and difficulty to our final, although I will break more problems into steps, i.e. as "part a), part b), part c)" and not have so many true/false questions. (although these ones are good practice!) Your midterm will give you a better idea of the format of my exams.

1. Define the following terms.

   (a) (3 points) immersion

   > **Solution:** $f : X \to Y$ is an immersion if for every $x \in X$, $df_x : T_x(X) \Rightarrow T_{f(x)}(Y)$ is injective

   (b) (3 points) homotopic maps

   > **Solution:** $f, g : X \to Y$ are homotopic if there exists a smooth map (homotopy) $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

   (c) (3 points) $T_x(X)$, when X is a manifold with boundary and $x \in \partial X$

   > **Solution:** Let $\phi : U \subset \mathbb{H}^k \to V \subset X$ be a parametrization near x. Then, $T_x(X) = \operatorname{Im} d\Phi_{\phi^{-1}(x)}$ where Φ is an extension near $\phi^{-1}(x)$ of ϕ to an open subset of $\mathbb{R}^k$

   (d) (3 points) mod 2 winding number

   > **Solution:** For $f : X \to Y$, $W_2(f, z) = \deg_2(\frac{f(x)-z}{|f(x)-z|})$.

2. Determine whether the following statements are true or false. No justification is required.

(a) (3 points) Every k-dimensional manifold has an embedding into $\mathbb{R}^{2k+1}$.

TRUE

(b) (3 points) If $(X, \mathcal{T})$ is a connected topological space, then for any $x, y \in X$, there exists a continuous path from x to y.

FALSE

(c) (3 points) Homotopy is an equivalence relation.

TRUE

(d) (3 points) If $f : S^k \to \mathbb{R}^{k+1}$ is a smooth map whose image does not contain the origin and $f(-x) = -f(x)$ for all $x \in S^k$, then $W_2(f, 0) = 1$.

TRUE

(e) (3 points) Every compact, connected smooth 1-manifold is diffeomorphic to S^1.

FALSE

(unless you specify that you are talking about manifolds without boundary, in which case the answer is true)

(f) (3 points) If f is a smooth map from the open unit ball $B^n \subset \mathbb{R}^n$ into itself, then there exists a point $x \in B^n$ such that $f(x) = x$.

FALSE

(g) (3 points) For any smooth map $f : X \to Y$, where X and Y have the same dimension, we have $\deg_2(f) = \deg(f) \bmod 2$.

TRUE

3. Compute the following. Explain your computations, but a rigorous proof is not required.

(a) (4 points) $\deg_2(f)$, $f : S^1 \to S^1$ is the identity

> **Solution:** $f : S^1 \to S^1$ is a diffeomorphism, so df_x is an isomorphism, and hence for any $y \in S^1$, y is a regular value of f. But $f^{-1}(y) = y$ has a single point, so $\deg_2(f) = 1$.

(b) (4 points) $\deg_2(f)$, $f : S^1 \to \mathbb{R}$ is the projection onto the x-axis

> **Solution:** The projection on to the x-axis misses the point 2, so $f^{-1}(2) = 0$. As 0 is not the image, it is trivially a regular value, so $\deg_2(f) = 0$.

(c) (4 points) $I_2(X, Z)$, where X and Z are circles of unit radius in $\mathbb{R}^2$, X is centered at $(-1, 0)$, and Z is centered at $(1, 0)$.

> **Solution:** X and Z do not intersect transversally, so one must deform X a little bit, say by translating to the left by ϵ, which makes the intersection empty. This gives $I_2(X, Z) = 0$.

4. (a) (8 points) Let $f : H^3 \to \mathbb{R}$ be given by $x^2 + y^2 + xz$. Show that $S = f^{-1}(1)$ is a manifold with boundary, and determine its boundary.

> **Solution:** Note that $df_{(x,y,z)} = [2x + z, 2y, x]$. If either x or y or non-zero, then the second or third column of $df_{(x,y,z)}$ is non-zero, so the derivative map is surjective onto $T_{f(x,yz)}(\mathbb{R}) = \mathbb{R}$. If both are zero, then $f(x, y, z) = 0$, so $f^{-1}(1)$ is empty. Hence, f is transversal to $\{1\}$.
>
> Now, ∂H^k is the set $\{(x, y, 0) : x, y \in \mathbb{R}\}$. In particular, $z = 0$. So $\partial f : \partial H^k \to \mathbb{R}$ is given by $\partial f(x, y) = x^2 + y^2$. We have that $d\partial f_{(x,y)} = [2x, 2y]$ which is again surjective if either x or y is non-zero. This is always the case for $(x, y) \in \partial f^{-1}(1)$, so ∂f is also transversal to $\{1\}$. By the theorem on p. 60, as $\{1\}$ is a zero dimensional manifold without boundary, $f^{-1}(1)$ is a manifold with boundary, and $\partial f^{-1}(1) = f^{-1}(1) \cap \partial H^k = \partial f^{-1}(1)$. But $\partial f^{-1}(1)$ is the set of points $(x, y, 0)$ such that $x^2 + y^2 = 1$, which is the unit circle in the xy-plane.

(b) (6 points) Let f be as in the previous question. Let $\mathbb{R}$ have the standard orientation, and H^3 the standard orientation as a subset of $\mathbb{R}^3$. Then S can be given a pre-image orientation. Write explicitly a positive basis for $T_x(S)$, where $x = (0,1,0)$.

For your reference: If $f : X \to Y$, $Z \subset Y$, $f(x) = y$ *and* $f^{-1}(Z) = S$, *we define pre-image orientation using the conventions*

$$N_x(S,X) \oplus T_x(S) = T_x(Y)$$

$$df(N_x(S,X)) \oplus T_y(Z) = T_y(Y)$$

Solution: Let $x = (0,1,0)$. $T_x(S)$ is the kernel of $df_x = (0,2,0)$, this is spanned by $e_1 = (1,0,0)$ and $e_3 = (0,0,1)$. The normal space $N_x(S,X)$ is the span of $e_2 = (0,1,0)$.

Now $df_x(e_2) = 2$, which is positive in the standard orientation on $\mathbb{R}$. Since $T_y(\{1\})$ is empty, we don't need to worry about the second equation. The first equation says that, if e_2, v, w is a positive basis for $T_x(H^3) = T_x(\mathbb{R}^3)$, then v, w is a positive basis for $T_x(S)$. Since e_2, e_3, e_1 is a positive basis because the determinant of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is 1, it follows that e_3, e_1 is a positive basis for $T_x(S)$.

5. (10 points) Show that $\mathbb{R}^2$ and the cylinder $S^1 \times \mathbb{R}$ are not diffeomorphic.

Solution: There are many ways to answer this question. Here is one possible approach by comparing intersection numbers:

Let X, Z be submanifolds of $\mathbb{R}^2$, with X diffeomorphic to S^1 and Z closed. Then, since S^1 has codimension 1 in $\mathbb{R}^2$, the Jordan-Brouwer separation theorem implies that $S^1 = \partial W$ for some compact manifold with boundary W.

By the Boundary Theorem, then $I_2(X, Z) = 0$ for any closed manifold Z.

Now fix $p \in S^1$ and take $X' = S^1 \times \{0\}$ and $Z' = \{p\} \times \mathbb{R}$ to be submanifolds of the cylinder $S^1 \times \mathbb{R}$. It is easy to see that X' and Z' are transversal at their only intersection point $(p, 0)$ since

$$T_{(p,0)}(S^1 \times \mathbb{R}) = T_p(S^1) \times T_0(\mathbb{R})$$

but we also have

$$\begin{aligned} T_{(p,0)}(S^1 \times \{0\}) &= T_p(S^1) \times \{0\} \\ T_{(p,0)}(\{p\} \times \mathbb{R}) &= \{0\} \times T_0(\mathbb{R}). \end{aligned}$$

Hence, $T_{(p,0)}(S^1 \times \mathbb{R}) = T_{(p,0)}(S^1 \times \{0\}) + T_{(p,0)}(\{p\} \times \mathbb{R})$.

As the transversal intersection contains one point, $I_2(X', Z') = 1$. But if $f : S^1 \times \mathbb{R} \to \mathbb{R}^2$ is a diffeomorphism, then by the lemma shown in class, $I_2(f(X'), f(Z')) = I_2(X', Z') = 1$. As X' is diffeomorphic to S^1, so is $f(X')$. This contradicts our earlier statement by letting $X = f(X')$ and $Z = f(Z')$. Hence, such a diffeomorphism f cannot exist.

6. (a) (10 points) Let $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ denote the non-zero real numbers. Show that the function $F : \mathbb{R}_* \times \mathbb{R}^3 \to \mathbb{R}^3$, given by $F(t, v) = p + tv$ where $p \in \mathbb{R}^3$ is fixed, is a submersion.

Solution: We can let $p = (p_1, p_2, p_3)$ and $v = (v_1, v_2, v_3)$, so that $F(t, v) = F(t, v_1, v_2, v_3) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)$. Then,

$$dF_{(t,v)} = \begin{bmatrix} v_1 & t & 0 & 0 \\ v_2 & 0 & t & 0 \\ v_3 & 0 & 0 & t \end{bmatrix}.$$

In particular, as $t \neq 0$, then $dF_{(t,v)}$ is surjective, as the last three columns form a basis for $\mathbb{R}^3$.

(b) (10 points) Fix $p \in \mathbb{R}^3 \setminus S^2$. Show that almost every line through p intersects $S^2 \subset \mathbb{R}^3$ transversally.

Solution: By part (a), $F \pitchfork Z$ for any submanifold $Z \subset \mathbb{R}^3$, as $\operatorname{Im} dF_{(t,v)} = T_{F(t,v)}(\mathbb{R}^3)$. By the transversality theorem, then $f_v(t) = F(t,v)$ is transversal to Z for almost every $v \in \mathbb{R}^3$.

But $f_v^{-1}(S^2)$ is precisely the set of points t such that $f(t) = p + tv \in S^2$, or in other words, the set of points at which the line $l = \{p + tv : t \in \mathbb{R}\}$ intersects S^2 (we can ignore $t = 0$ since $f(0) = p$ is not in S^2 by assumption). But $T_{p+tv}(l) = \operatorname{span} v = \operatorname{Im} d(f_v)_t$, so this means exactly that $l \pitchfork S^2$, as desired.

So almost every line through p intersects S^2 transversally.

7. (5 points) Let X be a smooth manifold with boundary, and $x \in \partial X$. Show that there exists a smooth non-negative function $f : U \to \mathbb{R}$ on an open subset $U \subset X$ containing x such that $f(z) = 0$ if and only if $z \in U \cap \partial X$. (Hint: Consider $X = H^k$.)

Solution: Let $\phi : V \subset H^k \to U \subset X$ be a parametrization near x. Let x_k be the corresponding coordinate function, which gives for every $z \in U$ the k-th coordinate of $\phi^{-1}(z)$.

We can see that $x_k(z) = 0$ if and only if $z \in \partial U = U \cap \partial X$. Then set $f = x_k$ and U as above. U is an open subset of X by definition of parametrization.