A short proof that $\mathrm{Diff}_0(M)$ is perfect

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Abstract

In this note, we follow the strategy of Haller, Rybicki and Teichmann to give a short, self contained, and elementary proof that $\mathrm{Diff}_0(M)$ is a perfect group, given a theorem of Herman on diffeomorphisms of the circle.

1 Introduction

Let M be a compact manifold of dimension $n > 1$ and let $\mathrm{Diff}_0(M)$ denote the group of isotopically trivial diffeomorphisms of M. That $\mathrm{Diff}_0(M)$ is a perfect group was first proved by Thurston in [6]. Recently, Haller, Rybicki and Teichmann gave a fundamentally different proof in [2] and [4]. In fact, they prove a stronger form of "smooth perfection" and give bounds on commutator width of $\mathrm{Diff}_0(M)$ for some manifolds.

The purpose of this note is to show that if one only wants to show $\mathrm{Diff}_0(M)$ is perfect, then the techniques of Haller, Rybicki and Teichmann provide a remarkably simple proof. Our exposition follows the strategy of [2] (and an early version of [4], see [3]), but avoids discussion of the tame Frechet manifold structure on $\mathrm{Diff}_0(M)$ in favor of explicit formulae. As the perfectness of $\mathrm{Diff}_0(M)$ is widely cited, I thought it worthwhile to make available this short and widely accessible proof.

The proof we give here also applies to $\mathrm{Diff}_c(M)$, the group of diffeomorphisms of a possibly noncompact manifold that are supported on compact sets and isotopic to the identity through a compactly supported isotopy. We use only one deep theorem, a result of Herman on circle diffeomorphisms.

Theorem 1.1 (Herman [5])**.** There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_0(S^1)$ and a dense set of rotations R_θ by angles $\theta \in [0, 2\pi)$ such that any $g \in \mathcal{U}$ can be written as $R_\lambda[g_0, R_\theta]$ for some rotation R_λ and some $g_0 \in \mathrm{Diff}_0(S^1)$. Moreover, λ and g_0 can be chosen to vary smoothly in g.

Here $[g_0, R_\theta]$ denotes the commutator $g_0 R_\theta g_0^{-1} R_\theta^{-1}$. "Vary smoothly in g" can be made precise with reference to the Frechet structure on $\mathrm{Diff}_0(M)$, but for our purposes the reader may take it to mean the following.

Definition 1.2. A *smooth family* in $\mathrm{Diff}_0(M)$ is a family $\{g_t : t \in [0, 1]\}$ such that the map $(x, t) \mapsto (g_t(x), t)$ is a smooth diffeomorphism of $M \times [0, 1]$. A map $\phi : \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ *varies smoothly* if it maps smooth families to smooth families.

A more general version of Herman's theorem for the n-torus is used in both Thurston's original proof and the Haller-Rybciki-Teichmann proof, though Haller, Rybciki and Teichmann state that their methods work using only Herman's theorem for S^1. This note provides the details.

2 Reduction to $M = \mathbb{R}^n$ and diffeomorphisms near identity

Our goal is to show the following.

Theorem 2.1. Let M be a smooth manifold. Then $\mathrm{Diff}_0(M)$ is perfect. In fact, any diffeomorphism g can be written $g = [g_1, f_1][g_2, f_2]...[g_r, f_r]$ where each f_i is the time one map of a vector field X_i on M.

To do so, it will be sufficient to consider the case of compactly supported diffeomorphisms on $M = \mathbb{R}^n$. (Recall that the *support* of a diffeomorphism g is the closure of the set $\{x \in M \mid g(x) \neq x\}$.) This reduction is due to the well-known fragmentation property:

Lemma 2.2 (Fragmentation)**.** Let $\{U_i\}$ be a finite open cover of M. Then any $g \in \mathrm{Diff}_0(M)$ can be written as a product $g_1 \circ g_2 \circ ... \circ g_n$ of diffeomorphisms where g_i is compactly supported in some element of $\{U_i\}$.

Proof. The proof is straightforward, for completeness we outline it here, following [1] Ch. 2. Let g_t be an isotopy from $g_0 = id$ to $g_1 = g$. By writing

$$g = g_{1/r} \circ (g_{1/r}^{-1} g_{2/r}) \circ ... \circ (g_{r-1/r}^{-1} g_1)$$

for r large, and working with each factor $g_{k-1/r}^{-1} g_{k/r}$, we may assume that g and g_t lie in an arbitrarily small neighborhood of the identity.

Take a partition of unity λ_i subordinate to $\{U_i\}$ and define $\mu_k := \sum_{i \leq k} \lambda_i$. Now define $\psi_k(x) := g_{\mu_k(x)}(x)$. This is a C^∞ map, and can be made as close to the identity as we like by taking g_t close to the identity, but it is not a priori invertible. However, being invertible with smooth inverse is an *open* condition, so being sufficiently close to the identity *implies* that ψ_k is a diffeomorphism. By definition, ψ_k agrees with ϕ_{k-1} outside of U_k, and hence $g = (\psi_0^{-1}\psi_1)(\psi_1^{-1}\psi_2)...(\psi_{n-1}^{-1}\psi_n)$ is the desired decomposition of g, with each diffeomorphism $\psi_{k-1}^{-1}\psi_k$ supported on U_k. $\square$

It is also sufficient to prove that some neighborhood of the identity in $\mathrm{Diff}_c(\mathbb{R}^n)$ is perfect, because any neighborhood of the identity generates $\mathrm{Diff}_c(\mathbb{R}^n)$. The strategy is to first prove perfectness of a neighborhood of the identity for S^1, move to $\mathbb{R}^2$, and then induct on dimension.

3 Proof for S^1 and diffeomorphisms preserving vertical lines

Perfectness of $\mathrm{Diff}_0(S^1)$ is an easy consequence of Herman's theorem and the fact that $\mathrm{PSL}(2, \mathbb{R})$ is perfect so any rotation can be written as a commutator.

Lemma 3.1 (Perfectness for S^1)**.** There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_0(S^1)_0$ and $f_1, ...f_4 \in \mathrm{Diff}_0(S^1)$ such that any $g \in \mathcal{U}$ can be written $g = [g_1, f_1]...[g_4, f_4]$, with g_i depending smoothly on g. Moreover, f_i can be taken to be the time one map of a vector field on S^1.

Proof. Let $\mathcal{U}$ be as in Herman's theorem and let $g \in \mathcal{U}$. Then g can be written as $R_\lambda[g_0, R_\theta]$ with λ and g_0 depending smoothly on g. Let $f_4 = R_\theta$ (this is indeed the time one map of a vector field). Now we need only show that there exist vector fields X_1, X_2, X_3 so that the rotation R_λ can be written as a product of commutators $[g_1, \exp(X_1)][g_2, \exp(X_2)][g_3, \exp(X_3)]$ with g_i depending smoothly on λ. We do this explicitly working in $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}_0(S^1)$, with Lie algebra of vector fields $\mathfrak{sl}(2,\mathbb{R})$.

Let

$$X_1 = X_3 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R}), \text{ and } X_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R}).$$

Define

$$B_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

.

Then $[B_\alpha, \exp(X_1)] = \begin{pmatrix} 1 & \alpha^2 - 1 \\ 0 & 1 \end{pmatrix}$ and $[B_\beta, \exp(X_2)] = \begin{pmatrix} 1 & 0 \\ \beta^{-2} - 1 & 1 \end{pmatrix}$.

Assume that $-1 \leq \beta \leq 1$, and $\beta^2 = \frac{\alpha^2}{2-\alpha^2}$. Then

$$[B_\alpha, \exp(X_1)][B_\beta, \exp(X_2)][B_\alpha, \exp(X_3)] = \begin{pmatrix} 1 + (\alpha^2 - 1)(\beta^{-2} - 1) & -\beta^{-2} + 1 \\ \beta^{-2} - 1 & 1 + (\alpha^2 - 1)(\beta^{-2} - 1) \end{pmatrix}$$

and this is a rotation by $\sin^{-1}(-\beta^{-2} + 1)$. This shows that a rotation can be written in the desired form. □

As an easy consequence, we now prove a perfectness result for compactly supported diffeomorphisms of $\mathbb{R}^n$ that preserve vertical lines:

Proposition 3.2. Let $U \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ be precompact. There exist vector fields $Y_1, ...Y_4$ supported on a neighborhood of U such that any diffeomorphism g supported on U that preserves vertical lines and is sufficiently close to the identity can be written as a product of commutators $[g_1, \exp(Y_1)]...[g_4, \exp(Y_4)]$ with g_i depending smoothly on g.

Proof. Let B be a ball in $\mathbb{R}^{n-1}$. There exists an embedding ϕ of $S^1 \times B$ in $\mathbb{R}^n$ containing U and contained in any small neighborhood of U such that for each $b \in B$, the image $\phi(S^1 \times \{b\}) \cap U$ is a vertical line segment as in Figure 1.

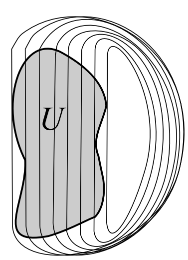

Figure 1: An embedding of $S^1 \times B$, vertically foliated on U, for $B = [0, 1]$

If g preserves vertical lines, then we can consider it as a map $\mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}$ of the form $(x, y) \mapsto (x, \hat{g}(x, y))$. For each $x \in \mathbb{R}^{n-1}$ let $g_x(y)$ denote $\hat{g}(x, y)$. Then g_x has support on a vertical line in U so we can consider it as a diffeomorphism of S^1 by pulling it back to $S^1 \times \{b\}$ via ϕ. Using Lemma 3.1, write $\phi^*(g_x) = [g_{x,1}, \exp(X_1)]...[g_{x,4}, \exp(X_4)]$. Push the vector fields X_i on each $S^1 \times \{b\}$ forward to $\mathbb{R}^n$ to get vector fields on $\phi(S^1 \times B)$ tangent to $\phi(S^1 \times \{b\})$ and extend these smoothly to vector fields Y_i with support in a small neighborhood of U. The smooth dependence of $g_{x,i}$ on g_x and hence on x means that the functions $\phi g_{x,i} \phi^{-1}$ on the vertical lines $\phi(S^1 \times \{b\})$ piece together to form smooth functions g_i supported on a neighborhood of U. By construction $g = [g_1, \exp(Y_1)]...[g_4, \exp(Y_4)]$. □

4 Proof for $\mathbb{R}^n$

The proof of Theorem 2.1 for $\mathbb{R}^n$ will follow from a short inductive argument using Proposition 3.2 and the following lemma:

Lemma 4.1. *There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_c^\infty(\mathbb{R}^n)_0$ such that any $f \in \mathcal{U}$ can be written as $g \circ h$ where h preserves each vertical line and g preserves each horizontal hyperplane, i.e. for $x = (x_1, ..., x_{n-1})$, we have $h(x, y) = (x, \hat{h}(x, y))$ and $g(x, y) = (\hat{g}(x, y), y)$. Moreover, g and h can be chosen to depend smoothly on f.*

Proof. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denote projection to the i^{th} coordinate. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is compactly supported and sufficiently C^∞ close to the identity, then for any point $(x, y) = (x_1, ..., x_{n-1}, y)$ the map $f_x : \mathbb{R} \to \mathbb{R}$ given by $f_x(y) = \pi_n f(x, y)$ is a diffeomorphism. (Injectivity follows from the fact that tangent vectors to vertical lines remain nearly vertical under a diffeomorphism close to the identity – if $\pi_n f(x, y_1) = \pi_n f(x, y_2)$ for some $y_1 \neq y_2$, then the image of f_x has horizontal tangent at some point $y \in [y_1, y_2]$.)

Now given f, define h and $g : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}$ by

$$h(x, y) = (x, f_x(y)), \text{ and}$$

$$g(x, y) = (g_1(x, y), ...g_{n-1}(x, y), y)$$

where $g_i(x, y) = \pi_i(x, f_x^{-1}(y)) \in \mathbb{R}$. Then $f = g \circ h$ and g and h vary smoothly with f. □

Proof of Theorem 2.1. We induct on the dimension n. The case $n = 2$ follows immediately from Lemma 4.1 for $n = 2$ and Proposition 3.2 applied to g and h in the decomposition (the proposition works just as well for g preserving horizontal lines). Now suppose Theorem 2.1 holds for $n = k$, and let $f \in \mathrm{Diff}_c^\infty(\mathbb{R}^{k+1})_0$ be close to the identity. By Lemma 4.1, $f = g \circ h$, where h preserves each vertical line and g preserves each horizontal hyperplane in $\mathbb{R}^{k+1}$, and g and h are close to the identity. By our inductive assumption, there are smooth vector fields $X_1, ..., X_{r(k)}$ tangent to each horizontal hyperplane – our hypothesis implies that these are defined on each $\mathbb{R}^k$- hyperplane, but the proof of Proposition 3.2 allows us to choose them so that they form a global vector field on $\mathbb{R}^{k+1}$ – and such that $g = [g_1, \exp(X_1)]...[g_r, \exp(X_{r(k)})]$ where the g_i preserve horizontal hyperplanes as well. By Proposition 3.2, there are also vector fields $Y_1, ..., Y_4$ supported on a neighborhood of $\mathrm{supp}(h)$ so that $h = [h_1, \exp(Y_1)]...[h_4, \exp(Y_4)]$. Thus, $f = g \circ h$ is a product of commutators as desired. □

References

[1] A. Banyaga *The Structure of Classical Diffeomorphism Groups*. Mathematics and its Applications, Kluwer Acad. Publ. (1997).

[2] S. Haller, J. Teichmann *Smooth Perfectness through Decomposition of Diffeomorphisms into Fiber Preserving Ones*. Ann. Glob. Anal. and Geom. 23 no. 1 (2003) 53–63

[3] S. Haller, J. Teichmann *Smooth perfectness for the group of diffeomorphisms*. Preprint. http://arxiv.org/abs/math/0409605v1 (version 1)

[4] S. Haller, T. Rybicki , J. Teichmann *Smooth perfectness for the group of diffeomorphisms*. Preprint. http://arxiv.org/abs/math/0409605v3

[5] M. Herman *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations* Pub. Math. IHES 49 (1979) 5–233.

[6] W. Thurston *Foliations and groups of diffeomorphisms*. Bull. Amer. Math. Soc. 80(1974), 304-307.