ON THE HOMOLOGY OF LIE GROUPS MADE DISCRETE

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Our goal today is to compare the (co)homology of BG and BG^δ, for G a Lie group. Here BG denotes the classifying space of G as a topological group, while BG^δ denotes its classifying space as a discrete group. The situation is fundamentally different for real (or rational) and finite coefficients. For characteristic zero, the comparison map is essentially always zero, while in the finite coefficient case it is expected to an isomorphism. Milnor proved several theorems making this precise, which we will discuss in this lecture. *We will closely follow Milnor's paper* [Mil83]*, and really you should be reading his paper instead of these notes.*

1. Compact Lie groups and Chern-Weil theory

We start with Lemma 8 from the Appendix of [Mil83].

Theorem 1.1. *If G is a compact Lie group, then*

$$H^i(BG;\mathbb{R}) \to H^i(BG^\delta;\mathbb{R})$$

is zero for $i > 0$.

Proof. This is a direct consequence of Chern-Weil theory, which says that the G-invariant (under the adjoint action) polynomials with real coefficients on $\mathfrak{g}$, denoted $I(G)$ by us, gives rise to all real cohomology of BG. The map $I(G) \to H^*(BG;\mathbb{R})$ is defined as follows. Given a smooth principal G-bundle $P \to B$ on M, recall that a connection on P is given by a G-invariant $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P,\mathfrak{g})$ such that $\omega(\frac{d}{dt}\exp(t\xi)) = \xi$ for all $\xi \in \mathfrak{g}$. In other words the form ω is essentially the projection map $TP \to T_vP$, the latter denoting the vertical vectors, and its kernel is the horizontal subbundle that appears in Ehresmann's definition of a connection on a principal G-bundle. Out of this we can define a curvature by noting that $d\omega + \frac{1}{2}[\omega,\omega]$ (a bracket of forms is given by wedging to get a form with values in $\mathfrak{g}\wedge\mathfrak{g}$ and then applying the Lie bracket to the coefficients, which makes sense since the Lie bracket is anti-symmetric) is a closed, but not necessarily exact form which is the pullback of some $\Omega \in \Omega^2(B,\mathfrak{g})$. Applying an element f of $I(G)_k$ to the coefficients of $\Omega^k \in \Omega^{2k}(B,\mathfrak{g}^{\otimes k})$, we get $f(\Omega) \in \Omega^{2k}(B,\mathbb{R})$.

In the universal case we get a map

$$I(G) \to H^{2*}(BG,\mathbb{R})$$

which is an isomorphism. The result now follows easily. To show that $[f(\Omega)]$ is zero it suffices to show it evaluates to zero on each real homology class of BG^δ. Any real homology class a of BG^δ is the image of a class $\tilde{a}$ in $H_*(X;\mathbb{R})$ for some manifold X (a is represented by a cycle contained in some finite subcomplex of BG^δ and we can take X to be a regular neighborhood of this subcomplex in some Euclidean space), so it suffices to evaluate $[f(\Omega)]$ on $\tilde{a}$. But a G^δ-principal bundle is a flat G-bundle, and this implies that $\Omega = 0$, so that $[f(\Omega)](\tilde{a}) = 0$ as long as f is of positive degree. $\square$

Chern-Weil theory also work if G is complex, as long as it is semi-simple and has finitely many components (see Lemma 12 of the Appendix of [Mil83]).

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2. General Lie groups and van Est theory

What happens if G is not compact? In this case Chern-Weil theory does not work. For example, if $G = \mathbb{R}$ then its cohomology is of course trivial, while the polynomials on $\mathbb{R}$ (the adjoint action is trivial) are non-zero in all degrees. In this case van Est theory replaces Chern-Weil theory. Intuitively this cohomology keeps track of what happens in the non-compact directions.

Theorem 2.1. *If G is a Lie group with finitely many components and $K \subset G$ is a maximal compact subgroup, then*

$$H^i(BG;\mathbb{R}) \to H^i(BG^\delta;\mathbb{R})$$

is zero for $i > \dim(G/K)$.

Proof. Any universal principal G-bundle $EG \to BG$ can be approximated by manifolds $EG(N) \to BG(N)$ where $EG(n)$ is N-connected. Take $N > i$ and factor the map we are interested in as:

$$H^i(BG(N);\mathbb{R}) \to H^i(\Omega^*(EG(N))^G) \to H^i(BG^\delta;\mathbb{R})$$

where $\Omega^*(-)$ is the complex of smooth differential forms and $(-)^G$ takes the G-invariants. Note that since G is non-compact, one can not average and it is not the case that the complex $\Omega^*(EG(N))^G$ is isomorphic to the forms on $BG(N)$. For the first map, note that $H^i(BG(N);\mathbb{R})$ is computed by the cohomology of the complex of de Rham forms $\Omega^*(BG(N);\mathbb{R})$. Pulling back along the projection $EG(N) \to BG(N)$ gives a G-invariant form on $EG(N)$ (but recall not all G-invariant forms on $EG(N)$ are obtained this way, only the horizontal ones):

$$\Omega^*(BG(N);\mathbb{R}) \ni \omega \mapsto \pi_N^*\omega \in \Omega^*(BEG(N);\mathbb{R})^G$$

For the second map, consider the smooth singular complex $\mathrm{Sing}^{\mathrm{sm}}(E(N))$, which is highly-connected with a proper free G^δ-action, so that $\mathrm{Sing}^{\mathrm{sm}}(E(N))/G^\delta$ is a highly-connected approximation to BG^δ. Integrating forms over smooth simplices gives a map

$$\Omega^*(BEG(N);\mathbb{R})^G \ni \alpha \mapsto \int_{(-)} \alpha \in \mathrm{Hom}(\mathrm{DK}(\mathrm{Sing}^{\mathrm{sm}}(EG(N))/G^\delta),\mathbb{R})$$

where DK is the Dold-Kan functor from simplicial sets to chain complexes.

To prove that the map is zero for $i > \dim(G/K)$, it now suffices to prove that $H^i(\Omega^*(EG(N))^G)$ vanishes in that range. To prove this, we use the techniques of van Est (see Chapter IX of [BW00]). The idea is that these cohomology groups can be identified with the smooth cohomology of G (i.e. use only smooth maps in the bar construction for cohomology), and these are the derived functors of $(-)^G = \mathrm{Hom}_G(\mathbb{R},-)$ applied to $\mathbb{R}$ in the category of Frechet modules with smooth G-action [BW00, IX.4.2] (where $\mathbb{R}$ has the trivial action). To see this, we use the general fact that if X is a manifold with proper smooth G-action, the complex

$$\mathbb{R} \to \Omega^0(X;\mathbb{R}) \to \Omega^1(X;\mathbb{R}) \to \ldots$$

is injective enough in this setting (Borel-Wallach call it s-injective, this is proven in [BW00, IX.5.4]). If X is highly-connected, the complex is highly acyclic, and this we obtain the identification we were looking for. Now note that G/K is always diffeomorphic to Euclidean space (in the case of $O(n) \subset \mathrm{GL}_n(\mathbb{R})$ this is the Gram-Schmidt procedure) and has a smooth and proper G-action. This means that smooth cohomology of G is also isomorphic to $H^i(\Omega^*(G/K;\mathbb{R})^G)$, which clearly vanishes for $i > \dim(G/K)$, because the complex is 0 in those degrees. Here it is important that the action of G on G/K is proper, which is for example not the case for $X = *$. □

We can do better if G is real semi-simple and connected. In that case Milnor proved that the kernel of $H^*(BG;\mathbb{R}) \to H^*(BG^\delta;\mathbb{R})$ in positive degrees is generated by the image of $H^*(BG_\mathbb{C};\mathbb{R})$ [Mil83, Theorem 2 of Appendix]

Example 2.2. If $G = \mathrm{SL}_n(\mathbb{R})$, $G_\mathbb{C} = \mathrm{SL}_n(\mathbb{C})$ and their cohomologies differ only that the former has an Euler class e (satisfying $e^2 = p_n$) while the latter does not. This means that the Euler class survives to $H^*(B\mathrm{SL}_n(\mathbb{R})^\delta;\mathbb{R})$.

3. Reductions for finite coefficients

The main topic of Milnor's paper is the isomorphism conjecture.

Conjecture 3.1 (Isomorphism conjecture)**.** *For any Lie group G and any prime p the map*

$$H_i(BG^\delta; \mathbb{Z}/p\mathbb{Z}) \to H_i(BG; \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism.

In particular, he proved the following theorem:

Theorem 3.2. *The isomorphism conjecture holds for all connected Lie groups if and only if it holds for simply-connected simple connected Lie groups.*

Proof. For a connected Lie groups, the isomorphism conjecture is equivalent to $B\bar{G} = \mathrm{hofib}(BG^\delta \to BG)$ (based at the identity) having the mod p-homology of a point (since BG is simply-connected, we can deduce this from the Serre spectral sequence). Note that not only does $B\bar{G}$ depend only on the identity component G_0 of G, but also, if U_0 is the universal cover G_0, we have that $U_0 \to G_0$ induces a weak equivalence $B\bar{U}_0 \to B\bar{G}_0$ (use the long exact sequence for homotopy groups) In this sense $B\bar{G}$ only depends on the local structure near the identity, and thus really only on the Lie algebra.

The simplest case is $G = \mathbb{R}$ with addition. Then $BG \simeq *$, so we have to prove that BG^δ has the mod p-homology of a point. As a discrete group $\mathbb{R} \cong \bigoplus \mathbb{Q}$ (an uncountable vector space over $\mathbb{Q}$), and by a colimit argument to prove the mod p-homology of finite-dimensional vector spaces over $\mathbb{Q}$ is trivial. By Künneth it suffices to prove this for $\mathbb{Q}$. But $\mathbb{Q}$ is a colimit of free cyclic groups generated by $1/N$: then clearly H_i for $i > 1$ vanishes and H_1 is trivial because in the colimit we end up inverting in particular the number p.

One next proves the result for solvable groups. We can assume these are given by a subgroup of the upper-triangular matrices obtained by taking all matrices fixing a flag. Using Künneth one reduces to case of all upper-triangular matrices G. Evaluating at an entry in the line just above the diagonal gives a homomorphism $G \to \mathbb{R}$ with kernel N a solvable groups of lower dimensional. Inductively, $B\bar{N}$ has trivial mod p-homology and now a Serre spectral sequence finishes the job.

Finally, for an arbitrary Lie group G the Lie algebra $\mathfrak{g}$ has maximal solvable ideal $\mathfrak{n}$ and $\mathfrak{g}/\mathfrak{n}$ is a direct sum of simple Lie algebra $\mathfrak{s}$. Though there might not be a fibration $N \to G \to \prod_i S_i$, there is a fibration $B\bar{N} \to B\bar{G} \to \prod_i B\bar{S}_i$ (since $B\bar{G}$ only depends on the Lie algebra). Another Serre spectral sequence combined with Künneth gives the final reduction to simple Lie groups. □

4. Split surjection for finite coefficients

Theorem 4.1. *If G has finitely many components, then*

$$H_i(BG^\delta; \mathbb{Z}/p\mathbb{Z}) \to H_i(BG; \mathbb{Z}/p\mathbb{Z})$$

is a split surjection.

Proof. Let K be a maximal compact subgroup, T a maximal torus in K and N its normalizer, then the universal cover of the identity component of N is solvable (the group of component of N is the Weyl group and the identity component is isomorphic to T). It is apparently also true that K/N has Euler characteristic $+1$ (one can prove this e.g. using Atiyah-Bott localization). Now consider the diagram

$$\begin{tikzcd}
H_*(BN^\delta; \mathbb{Z}/p\mathbb{Z}) \arrow[rr] \arrow[d, "\cong"'] & & H_*(BG^\delta; \mathbb{Z}/p\mathbb{Z}) \arrow[d] \\
H_*(BN; \mathbb{Z}/p\mathbb{Z}) \arrow[r, "i_*"] & H_*(BK; \mathbb{Z}/p\mathbb{Z}) \arrow[r, "\cong"'] & H_*(BG; \mathbb{Z}/p\mathbb{Z})
\end{tikzcd}$$

where all maps are the obvious ones. Now we note the existence of the Becker-Gottlieb transfer $i^!$ for the fibration $BN \to BK$ with fiber K/N of finite type [BG76]. This satisfies $i_* \circ i^! = \chi(K/N)\mathrm{id}$, so that i_* is split surjective. We conclude that the left-bottom composite is split surjective, and thus the right vertical arrow is split surjective as well. □

References

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