Godbillon-Vey Class II

Recap of last time: Bott Vanishing says that Pont. classes of $\underbrace{TM/E}_{\text{Normal bundle of } \mathcal{F}}$ vanish when $\underbrace{E \text{ is an integrable distribution}}_{\mathcal{F} \text{ is a foliation on } M}$ (equivalently: Normal bundle of $\mathcal{F}$ ... $\mathcal{F}$ is a foliation on M)

What are characteristic classes of foliations?
"when one obstruction vanishes, another appears ..."

We defined Godbillon-Vey class $GV(\mathcal{F}) \in H^3(M;\mathbb{R})$ as follows:
Let $\mathcal{F}$ be codim-1 foliation. Then $\mathcal{F} = \ker(\omega)$ for 1-form ω.
$d\omega = \omega \wedge \theta$ for $\theta \in \Omega^1(M)$ (since $\ker(\omega)$ integrable)
$GV(\mathcal{F}) := \theta \wedge d\theta \in H^3(M;\mathbb{R})$.

Can do the same thing for higher codimension: $\omega \in \Omega^q(M)$ defining $\mathcal{F} = \ker(\omega)$ codim q foliation. Then $d\omega = \omega \wedge \theta$, $\theta \in \Omega^1(M)$
Define $GV(\mathcal{F}) = (d\theta)^q \wedge \theta \in H^{2q+1}(M;\mathbb{R})$.

Closed since $0 = d(\omega\wedge\theta) = \cancel{d\omega \wedge \theta} + \omega \wedge d\theta$
$\Rightarrow d\theta \in$ ideal of forms vanishing on every leaf of $\mathcal{F}$
$\Rightarrow (d\theta)^{q+1} = 0$ (take basis for ideal)
$\Rightarrow d((d\theta)^q \wedge \theta) = (d\theta)^{q+1} = 0$.

Also independent of choice of ω, θ (tedious to check).

Pull back: if $\mathcal{F}_2$ is a foliation on M_2 and $f: M_1 \to M_2$ pulls back $\mathcal{F}_2$ to a foliation $\mathcal{F}_1$ on M_1 (i.e. f is transverse to $\mathcal{F}_2$), then $f^* GV(\mathcal{F}_2) = GV(\mathcal{F}_1)$. So GV should be a "characteristic class of foliations" and come from the cohomology of some classifying space (A universal foliated space?)
Technical problem: not all maps pull back foliations to foliations
Solution: Generalize foliation to Haefliger structure, build classifying space for Haefliger structures.

BUT FIRST:
- geometric meaning of GV class
- continuous variation of GV.

Helical Wobble: two perspectives

(1) Picture for Roussarie's foliation.

$SL_2\mathbb{R} = UT(\mathbb{H}^2)$.

Foliation by $\ker(\omega)$; ω dual to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in sl_2\mathbb{R}$

$\ker(\omega)$ spaned by (left-translats of) a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ = tangent vector to $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

flow along horocycle based at ∞

... ↑.↑.↑ ...

and b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

tanget vector to $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, vertical flow.

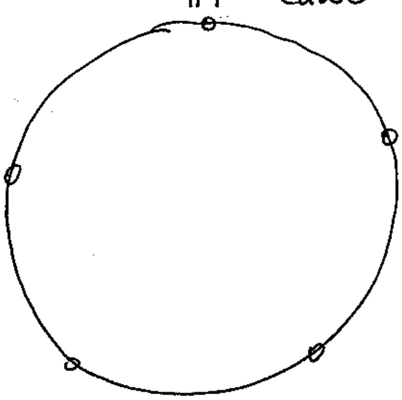
Leaves of foliation are vectors that point at same pt at infinity –

leaf through ↕ is all vertical tangent vectors.

Disc model: identify leaves with points of $S^1 = \partial\mathbb{H}^2$

let's pick 5 leaves and see how moving around

$\mathbb{H}^2$ causes them to get closer & further apart (idea in Ghys' paper)



(2) Reinhart – Wood's formula: $\mathcal{F}$ codim 1 foliation on Riemanian M^3.

• metric gives cannonical choice of G-V class, as follow:

Take curves γ normal to $\mathcal{F}$. k = curvature of normal curve

τ = torsion

$$GV = k^2 \cdot (\tau + \mathrm{II}(N, Z)) \cdot \mathrm{vol}$$

Normal field to γ — orthog. to N, tangent to leaf, N, Z, γ' form orthonormal frame

2nd fnd. form of leaf

Similar formula for codim-1 foliation on M^n, $n > 3$

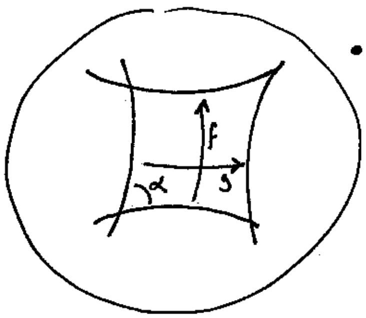
Continuous Variation of GV

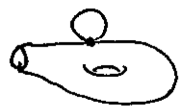
Thm (Thurston): $\exists$ continuous family of foliations on S^3 with ctsly varying GV class.
Can construct foliations s.t. $GV(\mathcal{F})$ takes any value in $\mathbb{R}$.

Construction from Morita's book, for $M^3 = S^1$-bundle over Σ_2 (actually $S^1 \times \Sigma_2$) instead of S^3

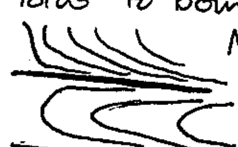
Recall: Roussarie foliation on $PSL_2\mathbb{R}$ descends to $PSL_2\mathbb{R}/\Gamma$ = Unit tangent bundle of hyperbolic surface. Σ_g
GV is volume form, so
$\langle GV(\mathcal{F}), [M^3] \rangle = vol(M^3) = area(\Sigma_g) = 2\pi \chi \Sigma_g$.
$\uparrow$ fund. class of $PSL_2\mathbb{R}/\Gamma$

Can build family of hyperbolic surfaces w/ geodesic boundary s.t. volume of unit tangent bundle varies continuously; as follow



- hyp. square with vertex angle α, $0 < \alpha < \frac{\pi}{2}$.
  - identify sides via hyp. isoms f and g; induces identification on U.T.; f, g preserve foliation so get foliated S^1-bundle over torus (w singularity at vertex)
  - so remove ε-nbd of vertex before gluing, get $M_1 =$ 
- $\partial M_1 = \mathbb{T}^2$. Foliation on ∂M_1 is linear of slope 4α.
  (can see this by identifying unit tangent circle over each point with S^1_∞)
- $Vol(M_1) = 2\pi - 4\alpha$ $-(\varepsilon$-ish amount$)$.
  If $H^3(M_1)$ was nonzero, we'd be done

Thurston's solution to "close" M_1: glue in solid torus to boundary. "Spin" foliation around boundary, foliate torus by Reeb foliation



GV(Reeb) = 0, so (hopefully) this works. I don't know about details.

Morita's book: different solution, gives details.

Sketch :

- do same construction to build  

call this M_2.
but then take K-fold fiberwise cover. Volume increases by k, $vol(M_2) = k(2\pi - 4\beta)$

If vertex angle of hyperbolic square is β, slope of linear foliation on ∂M_2 is $\frac{4\beta}{K}$. $\beta = k\alpha$ means slope matches M_1.

- Glue M_1 & M_2 along bdry, get foliated $\Sigma_2 \times S^1$. IF foliation was smooth (i.e. 1-form defining $\mathcal{F}$ on M_1 and 1-form defining $\mathcal{F}$ on M_2 made global 1-form)

Then $\langle GV(\mathcal{F}), [M_1 \cup_\partial M_2] \rangle = (2\pi - 4\alpha) - k(2\pi - \overset{k4\alpha}{\cancel{4\beta}}) = 2\pi(1-k) + 4(k^2-1)\alpha$

$\uparrow$ M_2 has opposite orientation $(\pm \varepsilon)$

- Show : • can approximate $\mathcal{F}$ by $\ker(\omega)$, ω smooth
  agrees with original except on tiny nbd of torus.
  - result was actually independent of ε-nbd of vertex removed at the beginning.

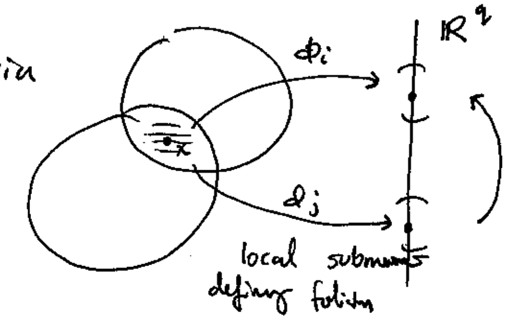
GV as a characteristic class of I foliations
II flat bundles
III foliated product bundles.

I. Haefliger's classifying space

$\Gamma_q = \Gamma_q^\infty$ groupoid of germs of smooth diffeos of $\mathbb{R}^q$

Foliation on M gives a Γ_q structure: maps $U_\alpha \cap U_\beta \to \Gamma_q$ satisfying 1-cocycle condition

via



germ of comparison map g_{ij}
$(\Phi_i = g_{ij}(x) \circ \Phi_j)$

$B\Gamma_q$ = classifying space for Haefliger structure.

Can extend GV to a class in $H^* B\Gamma_q$ ...

In this language, Bott vanishing says: the map $H^*(BGL_q\mathbb{R}; \mathbb{R}) \to H^*(B\Gamma_q; \mathbb{R})$ induced by codim q foliation $\to$ Normal bundle is the zero map in degrees $> 2q$.

II. Recall:

Flat F-bundle over X = bundle + codim $(\dim F)$ foliation transverse to the fibers.

Structure group = $\mathrm{Diff}(F)^\delta$

Classified by maps $X \to B\mathrm{Diff}(F)^\delta$.

Flat S^1 bundle over X gives codim 1 foliation on total space M.

get $GV(M) \in H^3(M; \mathbb{R})$.

Integrate along fiber to get class in $H^2(X; \mathbb{R})$

This procedure defines "gv" class in $H^2(B\mathrm{Diff}(S^1)^\delta; \mathbb{R})$

Since $H^*(B\mathrm{Diff}(S^1)^\delta ; \mathbb{R})$ is just group cohomology (bar construction)

should have cocycle ~~formula~~ $c : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \to \mathbb{R}$ representing gv.

<u>Thm</u>: (Thurston) gv is represented by

$$c(f, g) = \int_{S^1} \log(g') \, D \log(fg)' \, dt .$$

<u>III</u>. foliated products.

Recall, have map $B\mathrm{Diff}(S^1)^\delta \to B\mathrm{Diff}(S^1)$

homotopy fiber of this map is called $B\overline{\mathrm{Diff}(S^1)}$, and classifies "foliated S^1-products" foliated bundles that are trivial as bundles.

Since GV is an invariant of foliation, (it's believable that) it survives the map $H^*(B\mathrm{Diff}(S^1)^\delta) \to H^*(B\overline{\mathrm{Diff}}(S^1))$.

"The cohomology of $B\overline{\mathrm{Diff}}(S^1)$ is a Gelfand-Fuchs cohomology"

(there is a homomorphism $H^*_{GF}(S^1) \to H^* B\overline{\mathrm{Diff}}(S^1)$)

Our next goal is to learn GF cohomology (from Bott's perspective) and see natural setting for char. classes including GV class.