§1. Topological obstruction to integrability (Bott's vanishing)

Def A subbdle of a tangent bdle is called integrable if its space of sections is closed under bracket.

Question Is every subbdle of a tangent bdle deformable into an integrable one?

Motivation: $\begin{array}{c} Y \\ \downarrow \pi \\ X \end{array}$ is a C^∞-fibering and $E \subset TY$ is tangent along the fibers $E = T_\pi Y$

$$0 \longrightarrow E \longrightarrow TY \longrightarrow \pi^* TX \longrightarrow 0$$

$$\Longrightarrow \quad \mathrm{Pont}^k\left(TY/E \right) = 0 \quad \text{as long as} \quad k > \dim X$$

Pontryagin ring of deg k

Thm (Bott's vanishing) Let $E \subset TM$ be an integrable subbdle of codim q. then $\mathrm{Pont}^j\left(TM/E \right) = 0$ for $j > 2q$.

Pf, Boils down to two facts

1) Chern-Weil theory 2) TM/E has a particular connection whose curvature ~~form~~ tensor raised to powers $> q$ is zero.

So consider $Q = TM/E$, its dual $Q^* \subset TM^*$ ← annihilating E.

if $\omega_1, \dots, \omega_q$ generate $\Gamma(Q^*|_U)$ (sections over U)

$\Longrightarrow$ (Frobenius integrability) $\exists$ 1-forms $\theta_{ij} \in \Omega^1_U(M)$ s.t.

$$d\omega_i = \sum_{j=1}^{q} \theta_{ij} \wedge \omega_j \qquad 1 \le i \le q$$

i.e. $\Gamma(Q^*) \subseteq \Omega^1(M)$ generates a differential ideal in $\Omega^*(M)$

How to define a connection D_U on $Q^*|_U$? We can assign to $D_U(\omega_i)$ any q sections $s_1, \dots, s_q \in \Gamma(TM^* \times Q^*|_U)$

i) extend D_U additively ii) for any smooth $f: U \to \mathbb{R}$

$$D_U f\omega_i = f D_U \omega_i + df \otimes D\omega_i$$

Define : $$D_U(\omega_i) = \sum_{j=1}^{q} \theta_{ij} \otimes \omega_j$$

for any open set $U' \subset M$ $\omega_1', \dots, \omega_q'$ basis for $\Gamma(Q^*|_{U'})$

the image $D_{U'}(\omega_i')$ in $\Omega^2(M)$ coincide with $d\omega_i'$.

$$d\omega_i = \sum \theta_{ij} \wedge \omega_j \xrightarrow{d} 0 = \sum_j d\theta_{ij} \wedge \omega_j - \sum_j \theta_{ij} \wedge \left(\sum_k \theta_{jk} \wedge \omega_k\right)$$

$$= \sum_k \underbrace{\left(d\theta_{ik} - \sum_j \theta_{ij} \wedge \theta_{jk}\right)}_{\shortparallel} \wedge \omega_k$$

K_{ik} is (i,k) entry in curvature matrix

$$(*) \quad \sum_k K_{ik} \wedge \omega_k = 0$$

let $Z_1, \dots, Z_q$ dual to ω_i and let $X, Y \in \Gamma(E|_U)$

Apply (X, Y, Z_ℓ) to $(*) \Longrightarrow K_{i\ell}(X, Y) = 0$

$K_{i\ell}$ is in ideal $\langle \omega_1, \ldots, \omega_q \rangle$ if ϕ is an inv poly of

deg $> q$ $\qquad \phi((K_{i\ell})) = 0$ $\qquad \square$

Cor (Complex analytic case) A holomorphic sub bdle E of a holomorphic tangent bdle TM

$$\mathrm{Chern}^k(TM/E) = 0 \qquad k > \dim_{\mathbb{R}}(TM/E)$$

Cor. Let M be compact complex analytic mfld. which admits a non-vanishing holomorphic vector field. Then all the chern-numbers of M vanish.

Pf: E be the trivial sub bdle generated by the vector field.

it's obviously integrable $\mathrm{Chern}^{2n}(TM/E) = 0$ but since E is trivial

$$\mathrm{Chern}^{2n}(TM/E) = \mathrm{Chern}^{2n}(TM).$$

Cor. For n odd. there exists a holomorphic sub bdle E of

Complex codim 1. of $T\mathbb{C}P^n$ such that

$$0 \longrightarrow E \longrightarrow T\mathbb{C}P^n \longrightarrow H^2 \longrightarrow 0$$

H is Hyperplane bdle.

$$c_1\left(\frac{T\mathbb{C}P^n}{E}\right)^2 \neq 0 \implies \text{So } E \text{ is not integrable.}$$

Rm. let $\lambda : E \longrightarrow TM$ be an injection outside of the submfld

$\Sigma \subset X$ let its image be integrable there. Then for $\phi \in \mathrm{Chern}^k(TM/E)$

with $k > 2\dim_{\mathbb{C}}(TM/E)$ ~~can be written as~~ exists index theorem around singularity at Σ.

Cor (Shulman) If E is integrable and $a, b, c \in \mathrm{Pont}^*(TM/E)$

s.t. $\deg(a) + \deg(b) > 2q$ $\quad \deg(b) + \deg(c) > 2q$.

then Massey prod $(a,b,c) = 0$.

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§2 Bott's vanishing does not hold integrally:

μ_n ~~be~~ be n-th roots of unity.

$$M = \frac{S^{2k-1} \times \mathbb{R}^2}{\mu_n}$$, Now on $S^{2k-1} \times \mathbb{R}^2$ there exists a 2-form $\tilde{\omega}$

that restricts to $dxdy$ on $\{z\} \times \mathbb{R}^2$ for any $z \in S^{2k-1}$.

$\tilde{\omega}$ descends to M $\quad d\omega = 0$.

M is standard flat oriented two plane bdle over the lens space

$$X = \frac{S^{2k-1}}{\mu_n}, \quad \pi : M \longrightarrow X$$

$\mathcal{F}$ foliation on M

$\nu(\mathcal{F})$ = normal bdle of $\mathcal{F}$ = $\pi^*(M)$

$$p_1(\pi^*(M)) = \pi^*(p_1(M)) \qquad p_1(M)^2 \in H^8(X, \mathbb{Z})$$

$$H^{ev}(X, \mathbb{Z}) = \mathbb{Z}/n[x] \big/ x^k, \quad \deg x = 2. \qquad p_1(M) = x^2.$$

$k \geq 5$

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§3. Godbillon-Vey classes: $E \subset TM$ codim 1. suppose TM/E is trivial.

integrable.

$\omega : TM \to \mathbb{R}$ satisfies $d\omega = \theta \wedge \omega$ (Frobenius) for some $\theta \in \Omega^1(M)$.

Thm $\theta \wedge d\theta$ is a well-defined closed form in $H^3(M;\mathbb{R})$.

Pf: $d\theta$ is curvature. (Bott vanishing) $\Longrightarrow$ $d\theta \wedge d\theta = 0$

$\Longrightarrow$ $d(\theta \wedge d\theta) = 0$ so $\theta \wedge d\theta$ is closed.

ω is not unique

$\omega \rightsquigarrow f\omega$ for $f: M \to \mathbb{R}^*$

$$d(f\omega) = f \overset{\theta\wedge\omega}{d\omega} + df \wedge \omega = \underbrace{\left(\theta + \frac{df}{f}\right)}_{\xi} \wedge f\omega$$

$$\xi \wedge d\xi = \theta \wedge d\theta + \frac{df}{f} \wedge d\theta$$

is exact $\left(d\left(\frac{df}{f} \wedge \theta\right)\right)$.

$\theta \rightsquigarrow \theta + g\omega$

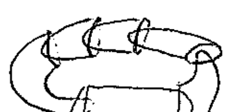
$d\omega = \theta \wedge \omega$

$d\theta \wedge \omega = 0$

$$(\theta + g\omega) \wedge (d\theta + g \overset{\theta\wedge\omega}{d\omega} + dg \wedge \omega) = \theta \wedge d\theta - dg \wedge \theta \wedge \omega = \theta \wedge d\theta - \underbrace{dg \wedge d\omega}_{\text{is exact}}.$$

Rm 1. $F\Omega_{n,1}$:= Cobordism class of mflds with Codim 1 foliations

GV-class is well-defined on $F\Omega_{n,1}$ i.e. it takes same value on cobordant foliations

Ex.  $GV(\text{Reeb foliation}) = 0$

Ex 1. Let L stand for $\mathbb{R}$ or S^1 and α for the standard 1-form on L. that is dt or "$d\theta$". If $M \to L$ is the proj of fiber bdle, then M is foliated by fibers of f. $\omega = f^*(\alpha)$ & $d\omega = 0$ so $GV_1 = 0$.

in a similar vein, if V is a mfld. $f: V \to \mathbb{R}$ smooth with 0 as a regular value. $V \times L$ has codim 1 foliation. its 1-form is $\omega = f\alpha + df$ no where zero and integrable. $d\omega = \omega \wedge \alpha$. $GV_1 = 0$.

What does GV measure?

$E \subset TM$

Let $X \in T(TM)$ w/ norm 1. in $E^{\perp}$. $\omega(X)=1$.

$L_X\omega = i_X d\omega + d i_X \omega = i_X d\omega \implies d\omega = L_X\omega \wedge \omega \xrightarrow{d} dL_X\omega\wedge\omega = 0$

$GV_1 = \{ L_X\omega \wedge dL_X\omega \}$ or $= \{\omega \wedge L_X\omega \wedge L_X^2\omega\}$

$\omega \wedge L_X\omega$ measures the tendency of a leaf to turn away from the previous nearby lines.

and $dL_X\omega = L_X(L_X\omega\wedge\omega)$ is the velocity in the normal direction with which leaves are turning away.

Thurston says $\omega\wedge L_X\omega\wedge L_X^2\omega$ measures sth like a gyroscopic procession. "helical wobble"

§4 <u>Non-vanishing example of Roussarie</u> ; $SL(2,\mathbb{R}) \supset \Gamma$ such that

$M = SL(2,\mathbb{R})/\Gamma$ is compact.

Foliation on M: Want to find ω on M such that $\omega\wedge d\omega = 0$. (Frobenius)

$sl(2,\mathbb{R})$: $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $[H,X] = 2X$, $[X,Y] = H$

$[X,Y] = -2Y$

$\updownarrow$ ξ, $\updownarrow$ η, θ

left inv 1-forms θ, ξ, η s.t. $d\theta = -\xi\wedge\eta$, $d\xi = -2\theta\wedge\xi$

$d\eta = 2\theta\wedge\eta$

$d\omega(X,Y) = -\omega([X,Y])$

We choose η to define the foliation.

$$d\eta = \underbrace{2\theta}\wedge\eta$$

$$2\theta \wedge 2d\theta = \underbrace{-4\,\theta\wedge\xi\wedge\eta}_{\text{vol form of the group.}} = -4\,\mathrm{vol}(M)$$

hence on M

Take very special case for Γ. Suppose S is a compact surface of genus ≥ 2.

$$\begin{array}{c} \mathbb{H} \\ \downarrow \\ S \end{array} \quad \text{universal cover}$$

$$\mathbb{H} = SL(2,\mathbb{R})/SO(2)$$

$$\pi_1(S)\backslash\mathbb{H} = S \simeq \underset{\substack{\uparrow \\ \text{extension} \\ \text{of } \pi_1(S) \text{ by } \mathbb{Z}/2}}{\pi}\backslash SL(2,\mathbb{R})/SO(2)$$

Hence $SL(2,\mathbb{R})/\pi$ is compact as well.

$$\begin{array}{ccc} SL(2,\mathbb{R})/\pi & \longrightarrow & \pi\backslash SL(2,\mathbb{R})/SO(2) \\ M & \longrightarrow & S \end{array}$$

tangent circle bdle over S. Moreover $\theta = \frac{1}{2}$ line element along the circles.

$$4\int_M \theta\wedge\xi\wedge\eta = 4\int_{SO(2)}\theta\int_S \xi\wedge\eta = 4\pi\ (\text{area of } S) = 4\pi^2(2-2g)$$

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§5 Application of Bott's vanishing to realization problems:

Question: does $\pi : \mathrm{Diff}^{\delta}(\Sigma_g) \longrightarrow \mathrm{Mcg}(\Sigma_g)$ admit right section?

Thm (Morita). $\pi^*: H^*(\mathrm{Mcg}(\Sigma_g);\mathbb{Q}) \longrightarrow H^*(\mathrm{Diff}^{\delta}(\Sigma_g);\mathbb{Q})$ $\quad \pi^*(k_i) = 0$ for $i \geq 3$, $g \geq 10$

for flat Σ_g-bdles

$$\begin{array}{ccc} \Sigma_g \to & E & \leftarrow \text{codim 2 foliation on } E. \\ & \downarrow p & \\ & M & \end{array}$$

$$e(T_pE)^4 = p_1(T_pE)^2 = 0$$
$$\downarrow$$
$$p_*(k_3)$$

So k_3 for flat Σ_g-bdles is zero.

Rm. Thurston proved that $H_*(\mathrm{Homeo}^\delta(\Sigma_g)) = H_*(\mathrm{Mcg}(\Sigma_g))$ so there is no cohomological obstruction to prove that there is no right section for $\mathrm{Homeo}^\delta(\Sigma_g) \longrightarrow \mathrm{Mcg}(\Sigma_g)$