Segal's model for delooping $B\mathrm{Diff}_c^\delta(\mathbb{R})$ (Mather's thm)

$\mathrm{Diff}_c^\delta(\mathbb{R})$ compactly supported diffeo of $\mathbb{R}$

Def Γ_n : Haefliger top Category

obj: $x \in \mathbb{R}^n$

Mor: germs of diffeomorphism

(Sheaf top)

$B\Gamma_n =$ classifying space of Haefliger category

Thm (Mather) $B\mathrm{Diff}_c^\delta(\mathbb{R}) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \Omega B\Gamma_1$

$G = \mathrm{Diff}_c^\delta(\mathbb{R})$

For $a \leq b$ let $G_{a,b} = \{ g \in G : \operatorname{supp} g \subset (a,b) \}$

Obs : $BG_{a,b} \hookrightarrow BG$ induces isomorphism in integer homology.

Def (delooping): obj: real numbers

$\mathrm{Mor}(a,b) = BG_{ab}$ $\qquad BG_{ab} \times BG_{b,c} \longrightarrow BG_{a,c}$

$B(BG) :=$ realization of the nerve of the category.

Thm 1 (Mather) $BG \xleftarrow{\overset{H_*}{\simeq}} \bullet \xrightarrow{\simeq} \Omega B(BG)$ is H_*-equiv

Thm 2 (Mather) $B(BG)$ has the homotopy type of $B\Gamma_1$

Segal's Model

Thm 1

$M(\mathbb{R}^n) :=$ discrete monoid of smooth embedding $\mathbb{R}^n$

$$BM(\mathbb{R}^n) \xleftarrow{\simeq} \cdot \xrightarrow{\simeq} B\Gamma_n$$

Thm 2 X is smooth mfld which is interior of a mfld. with boundary

then $\mathrm{Diff}^{\delta}(X) \xrightarrow{H_*\text{-eq}} M_a(X)$ is H_*-equiv.

discrete monoid of smooth emb, of X which are isotopic to diffeomorphism

Thm 3. if X is Compact mfld

$$B\,\mathrm{Diff}^{\delta}_c(X\times\mathbb{R}) \longrightarrow \Omega BM_0(X\times\mathbb{R})$$

is H_*-equiv ($M_0(X) \subset M(X)$ isotopic to id)

Rm on what $BM(X)$ classifies

$$[Y, BM(X)] = \left\{ \begin{array}{l} \text{Concordance classes of } Y' \xrightarrow{p} Y \text{ where} \\ \text{a) } Y' \text{ is smooth mfld w/ foliation} \\ \text{b) } p \text{ is smooth map} \\ \text{c) in a ngbd of each fiber } p: Y' \to Y \end{array} \right\}$$

the foliation is isomorphic to $Y\times X \to Y$ (over Y)

for $M_a(X)$ condition b) becomes smooth fiber bdle.

Homotopy theory of monoids: M is top monoid

$M \circlearrowright X$ acts

Def: htpy quotient $X/\!\!/M := \| A_\bullet \|$ where $A_p = X \times M^p$

$$\begin{tikzcd} X \arrow[r] & X/\!\!/M \arrow[d] \\ & BM \end{tikzcd}$$

Some properties: a) $*/\!\!/M = BM$, $M/\!\!/M = EM$ which is contractible

b) if $X \to X'$ is an M-map and htpy (homology) equiv then so is $X/\!\!/M \to X'/\!\!/M$

Note: $M_1 \backslash\!\!\backslash (X/\!\!/M_2) = (M_1 \backslash\!\!\backslash X)/\!\!/M_2$

Prop: A homomorphism of monoids $M \to M'$, $BM \to BM'$ is htpy (homology) equiv if $M'/\!\!/M$ is contractible (acyclic)

Proof: look at

$$\begin{tikzcd} M' \backslash\!\!\backslash M'/\!\!/M \arrow[r] \arrow[d] & M' \backslash\!\!\backslash M'/\!\!/M' \arrow[d] \\ BM \arrow[r] & BM' \end{tikzcd}$$

Group completion thm (Segal - McDuff)

$X/\!\!/_M \longrightarrow BM$ is a quasi-fibration (resp. homology fibration)

if each m in M acts on X by a htpy (resp hmlgy) equiv.

prop: If $\pi : M \longrightarrow Q$ is homomorphism of Monoids with kernel $\pi^{-1}(1) = K$,

$$BK \longrightarrow BM \longrightarrow BQ$$

is a htpy (resp H_*-) ~~equiv~~ fibration sequence providing

a) $M/\!\!/_K \longrightarrow Q$ is equiv

b) each q in Q acts by an equiv on $M \backslash\!\!\backslash Q$

pf. If $M/\!\!/_K \simeq Q$ then $M\backslash\!\!\backslash Q \sim M \backslash\!\!\backslash M /\!\!/_K \simeq BK$

$$\underset{BK}{M\backslash\!\!\backslash Q} \longrightarrow \underset{BM}{(M\backslash\!\!\backslash Q)/\!\!/_Q} \longrightarrow BQ$$

is htpy (H_*) fibration by GPT.

Rmk: if M is discrete

a category model for $X/\!\!/_M$:= category obj: X, $\mathrm{Mor}(x,x') = \{m \in M : xm = x'\}$

$= C(X, M)$

Prop $N \hookrightarrow M$ (discrete submonoid) w/ left cancellation, then $BN \xrightarrow{\simeq} BM$ if for any m_1, m_2 in M there is an m in M and n_1, n_2 in N such that $mn_1 = m_1$ and $mn_2 = m_2$.

Prop Suppose $1 \to K \to M \xrightarrow{\pi} Q \to 1$ is an exact sequence of discrete monoids, and that:

a) for each q in Q there is an m_q in $\pi^{-1}(q)$ such that $k \longmapsto m_q k$ is bij $K \to \pi^{-1}(q)$

b) for each q in Q the endomorphism $c_q : K \to K$ defined $km_q = m_q c_q(k)$ is a homotopy equiv (homology)

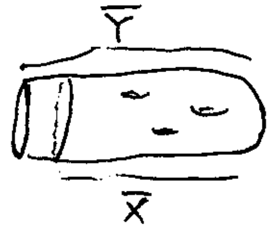
then $BK \to BM \to BQ$ is a htpy (homology) fibration seq.

proof: Want to check $q\times -$ induces an equiv $Q/\!/M$

$$\begin{tikzcd}
C(*,K) & */\!/K \arrow[r, "c_q"] \arrow[d, hook] \arrow[dr, phantom, "\Swarrow"] & */\!/K \arrow[d, hook] & \simeq BK \\
C(Q,M) & Q/\!/M \arrow[r, "q\times"] & Q/\!/M & \simeq BK
\end{tikzcd}$$

$\Swarrow$ is natural transformation $\times m_q$

Monoids of embeddings



$Y = \operatorname{int}(\overline{Y})$

$\overline{A} \cong \partial\overline{Y} \times [0,1)$, $X = \overline{Y} - \overline{A}$

Def: $M(Y,X) :=$ smooth embedding $\varphi : Y \to Y$, $\varphi(X) \subset X$

Prop.

$$M(Y) \xleftarrow{i} M(Y,X) \xrightarrow{r} M(X)$$

i, r induce htpy equiv.

proof. i is htpy equiv: let $\varphi_1, \varphi_2 \in M(Y)$

$\exists\, \varphi \in \operatorname{Aut}(Y)$ $\varphi(X) \supset \varphi_1(X) \cup \varphi_2(X)$

$\varphi_i = \varphi \circ (\varphi^{-1} \circ \varphi_i)$ $\varphi^{-1} \circ \varphi_i \in M(Y,X)$

Def: $\hat{M}(\overline{X}) :=$ germs of emb of $\overline{X}$ into itself. i.e. equiv classes

$\varphi : (U, \overline{X}) \to (Y, \overline{X})$

U is nbhd of $\overline{X}$ in Y

two such embs are equiv if they coincide in a nbhd of $\overline{X}$.

Prop: The homomorphism $\hat{M}(\overline{X}) \longrightarrow M(\overline{X}) \longrightarrow M(X)$ are htpy equiv.

Pf. $\hat{M}(\overline{Y}, \overline{X}) \xrightarrow{\simeq} \hat{M}(\overline{X})$ $M(\overline{Y}, \overline{X}) \xrightarrow{\simeq} M(\overline{X})$

$$\begin{tikzcd} M(Y,X) \arrow[r] \arrow[d] & M(X) \\ \hat{M}(\overline{X}) \arrow[r, "f"] & M(\overline{X}) \arrow[u, "g"] \end{tikzcd}$$

$\Longrightarrow$ this factorization implies f, g are htpy equivalences.

Def: $K(Y) :=$ Compactly supported diffeo of Y

$K(Y,X) := \{\varphi \in K(Y) : \varphi(X) \subset X\}$

Consider $K(Y,X) \xrightarrow{i} \hat{M}(\bar{X})$ with kernel $K(A)$ where

$A = Y - \bar{X}$.

Obs Istopy extension thm implies image of i is $\hat{M}_\partial(\bar{X})$ consisting of all φ in $\hat{M}(\bar{X})$ such that $\varphi|_{\partial X}$ is isotopic to id.

Prop a) $K(Y,X) \longrightarrow K(Y)$ is htpy equiv

b) The sequence $K(A) \longrightarrow K(Y,X) \longrightarrow M_\partial(\bar{X})$ is homology fib

Pf b) by the criteria we need to show for $\varphi \in K(Y,X)$

$c_\varphi : K(A) \longrightarrow K(A)$ $\quad c_\varphi(\psi) = \varphi^{-1}\psi\varphi$ is identity on H_*.

note that $\varphi^{-1}|_{\bar{A}} \in \hat{M}_\partial(\bar{A})$

Lemma: For any mfld $\bar{X}$ with bdry the conjugation action of $\hat{M}_\partial(\bar{X})$ on $K(X)$ $\quad (\varphi,\psi) \longrightarrow \varphi\psi\varphi^{-1}$ induces id on H_*.

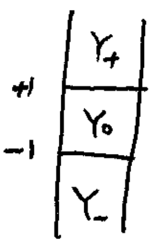
Pf: Conjugation by element of a group on cpt support

Thm if X is cpt manifold, there is a H_*-equiv

$$BK(X\times\mathbb{R}) \longrightarrow \Omega BM_0(X\times\mathbb{R})$$

where $M_0(X\times\mathbb{R})$ is the sub ~~[illegible]~~ monoid of $M(X\times\mathbb{R})$ which are isotopic to id.

proof:



$X\times\mathbb{R} = Y_+ \cup Y_0 \cup Y_-$ $\qquad Y_+ = X\times[1,+\infty)$, $\quad Y_0 = X\times(-1,1)$, $\quad Y_- = X\times(-\infty,-1]$

$$K(Y_0) \longrightarrow M(Y, Y_+; \mathrm{rel}\ Y_-) \longrightarrow \hat{M}_0(Y_+) \qquad (1)$$

id in a nghd of Y_-

$$\hat{M}_0(Y_+) \simeq M_0(\mathrm{interior}(Y_+)) \simeq M_0(X\times\mathbb{R})$$

their classifying space

it is enough to show a) $BM(Y, Y_+; \mathrm{rel}\ Y_-) \simeq *$ b) (1) is H_*-fibration

Pf a) $M(Y, Y_+; \mathrm{rel}\ Y_-) \longrightarrow M(Y; \mathrm{rel}\ Y_-)$ is htpy equiv.

filtering category

Pf b) if $\varphi_1, \varphi_2 \in M(Y, Y_+; \mathrm{rel}\ Y_-)$ have the same image in $\hat{M}(Y_+)$

$\Longrightarrow \varphi_1(Y) = \varphi_2(Y) \Longrightarrow \varphi_1^{-1}\circ\varphi_2 \in K(Y_0)$. so $\varphi_2 = \varphi_1(\varphi_1^{-1}\circ\varphi_2)$

(first condition of prop)

Second condition: $\varphi \in M(Y, Y_+; \mathrm{rel}\ Y_-) \Longrightarrow \varphi(\overline{Y_0}) \supset \overline{Y_0}$

$\varphi^{-1}|_{\overline{Y_0}}$ belongs to $M_0(\overline{Y_0})$