THE HOMOLOGY OF DIFFEOMORPHISM GROUPS

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Abstract. In this talk I'll give an overview of a successful approach to the homology of diffeomorphism groups: the complimentary techniques of homological stability and scanning. We will give an overview of known results and outline the techniques in the case of 0-dimensional manifolds.

Contents

1. Why do we care about the homology of diffeomorphism groups?

If G is a group, then one of the first steps to understanding G is computing its abelianization. If G is a topological group, you might instead start with computing the abelianization of its group of components. These abelian groups are isomorphic to $H_1(BG)$, the first homology group of the classifying space of G.

The first homology group is just the first in a list of invariants associated to G: the higher homology groups of BG. This talk is an introduction to modern methods for understanding the homology of diffeomorphism groups of manifolds, i.e. computing $H_*(B\mathrm{Diff}(M))$. We will discuss these methods and their applications later, but first explain why you should care about the homology of diffeomorphism groups.

Convention 1.1. Unless mentioned otherwise, all our groups are topological groups with their natural topologies. For example, Diff has the C^∞-topology, Homeo has the compact-open topology.

(i) *Homology is computable.* The first reason is a pragmatic one. The techniques developed for understanding the homology of diffeomorphism groups actually make it possible to do computations. This makes a starting point for further work. For example, Botvinnik, Ebert & Randal-Williams used our knowledge of $H_*(B\mathrm{Diff}(\#_g S^n \times S^n, D^{2n}))$ to understand spaces of positive scalar curvature metrics [BERW14] and Weiss used it to understand rational Pontryagin classes in $H^*(B\mathrm{Top}(2n);\mathbb{Q})$ [Wei15].

Example 1.2. As a concrete example, knowing that homology is non-zero allows one to disprove the existence of lifts or section. For example, if $\mathrm{Diff}(D^2\backslash k \text{ points}, \partial D^2)$ is the topological group of diffeomorphisms of the k-fold punctured two-dimensional disk fixing the boundary. There is a homomorphism $p : \mathrm{Diff}(D^2\backslash k \text{ points}, \partial D^2) \to \mathfrak{S}_k$ where the latter is the symmetric group, and our explicit knowledge of $H_*(B\mathrm{Diff}(D^2\backslash k \text{ points}, \partial D^2))$ and $H_*(B\mathfrak{S}_k)$ implies that there is no map $s : \mathfrak{S}_k \to \mathrm{Diff}(D^2\backslash k \text{ points}, \partial D^2)$ such that $p \circ s = \mathrm{id}$ if k is big enough; the homology of $\mathfrak{S}_k$ is simply too big in comparison to that of $\mathrm{Diff}(D^2\backslash k \text{ points}, \partial D^2)$ for the identity map to factor over $H_*(B\mathrm{Diff}(D^2\backslash k \text{ points}, \partial D^2))$ [CLM76].

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(ii) *Homology contains group-theoretic and homotopy-theoretic information.* Homology may seem a rather abstract invariant of groups at first, but our opening example shows that it is related to many concrete questions about groups. H_1 is the abelianization and H^2 contains information about central extensions. For example, Galatius & Randal-Williams used our knowledge of $H_1(B\mathrm{Diff}(\#_g(S^n \times S^n), D^{2n}))$ [GRW14b] to compute the abelianization of the mapping class group of $\#_g(S^n \times S^n)$, i.e. $\pi_0\mathrm{Diff}(\#_g(S^n \times S^n))$ [GRW14a].

On the other hand, homology contains homotopy-theoretic information. We know that the homotopy groups of $\mathrm{Diff}(D^n, \partial D^n)$ are finitely-generated in a range because Igusa [Igu88] was able to relate them in a range to Waldhausen's $A(*)$ [Wal85], which has shown to have finitely-generated homology by a twisted homological stability argument of Dwyer [Dwy80] and hence finitely-generated homotopy groups. In fact, Farrell-Hsiang computed them rationally in Igusa's range [FH78].

(iii) *Characteristic classes for manifold bundles.* The most direct application is not of the homology of diffeomorphism groups, but of their cohomology. The idea is that $B\mathrm{Diff}(M)$ classifies smooth manifold bundles with fibers M, just like $BO(n)$ classifies n-dimensional vector bundles. That is, there is so-called universal manifold bundle over $B\mathrm{Diff}(M)$ and pulling back this universal bundle gives a bijection:

$$\begin{array}{c}\{\text{homotopy classes of maps } X \to B\mathrm{Diff}(M)\}\\ \updownarrow \\ \{\text{isomorphism classes of smooth manifold bundles over } X \text{ with fiber } M\}\end{array}$$

If A is abelian group, an A-valued characteristic class c is a natural assignment of a cohomology class $c(E)$ in $H^*(X; A)$ to each manifold bundle E with fiber M over X. Naturality here means that given a map $f : X \to X'$, we have that $c(f^*E) = f^*c(E)$. Naturality implies all of these characteristic classes come from $B\mathrm{Diff}$ in the sense that pulling back along the classifying map gives a bijection

$$\begin{array}{c}\{A\text{-valued characteristic classes of manifold bundles with fiber } M\}\\ \updownarrow \\ \{H^*(B\mathrm{Diff}(M); A)\}\end{array}$$

Computing $H^*(BSO(n))$ is how we were able to show that all rational characteristic for oriented vector bundles are given by the Euler class and Pontryagin classes. For manifold bundles of even dimension the so-called stable rational classes have a nice geometric interpretation, being some type of fiberwise signatures. This is not the case for odd dimensions or finite field coefficients.

2. The stability/scanning partnership

The method to compute homology of diffeomorphism groups has two steps and will not be able to compute all the homology of all diffeomorphism groups. Instead, one fixes a class of manifolds and a stabilization construction sending M to $t(M)$ (often by adding a new part N to your manifold) such that it induces a map (often by extending diffeomorphisms by the identity on N)

$$t : \mathrm{Diff}(M) \to \mathrm{Diff}(t(M))$$

One then only expects to get information in a range, which increases as your manifold is in the image of more applications of t.

In this generality, we will have to be vague, but hopefully still inspiring. The first step is proving *homological stability*: proving that $t_* : H_*(B\mathrm{Diff}(t^k(M))) \to H_*(B\mathrm{Diff}(t^{k+1}(M)))$ is an isomorphism in a range. The idea is that many copies of N become available, so that even there is no global inverse to the stabilization map, there are locally many choices for an inverses and working in homology allows you to "glue" these. The idea is that one will prove that when allowing interpolations, the space of "inverses" is highly-connected.

The second aspect is computing the limit $\mathrm{colim}_{k\to\infty} H_*(B\mathrm{Diff}(t^k(M)))$. In practice this is often done by a technique called *scanning*. The idea is that in the limit the spaces involved become more "flexible" and homotopy-theoretic techniques apply. In particular, in favorable circumstances taking the limit is the same as inverting t, so that you are allowed to zoom in as long as things lost when zooming in are in the image of t.

Example 2.1. A homotopical example is $BSO(n)$ of this two-fold approach. The groups $SO(n)$ do not just satisfy homological stability, but even homotopical stability: there is a fibration $SO(n) \to SO(n+1) \to S^n$, and the long exact sequence of homotopy groups shows that $SO(n) \to SO(n+1)$ is at least $(n-2)$-connected. Thus one can obtain information about $BSO(n)$ by studying BSO. But this space is approachable by Bott periodicity: $\Omega^\infty BSO \simeq \mathbb{Z} \times BSO$, so that we only need to compute only eight homotopy groups of BSO to know all of them.

As an illustrative example I will tell you what is known in dimension 0, before sketching a proof making precise the vague statements made in the previous paragraph. All compact zero-dimensional manifolds are diffeomorphic to a finite number of points: $M \cong \sqcup_k *$ The diffeomorphisms of such a manifold are isomorphic to the symmetric group $\mathfrak{S}_k$. Taking the disjoint union with an additional point gives a map $\sqcup_k * \to \sqcup_{k+1} *$ which induces a homomorphism $t : \mathfrak{S}_k \to \mathfrak{S}_{k+1}$. Nakaoke proved that this map induces an isomorphism in a range [Nak60].

Theorem 2.2. *The map*

$$t_* : H_*(B\mathfrak{S}_k) \to H_*(B\mathfrak{S}_{k+1})$$

is an isomorphism for $* < \frac{k}{2}$ *and a surjection for* $* \leq \frac{k}{2}$.

In fact the map t_* is always injective, proven by a transfer argument. We will not prove this since it is not a general feature of homological stability results. The theorem implies that the homology of $H_*(B\mathfrak{S}_k)$ is independent of k for $*$ small. In particular, the stable homology $\mathrm{colim}_{k\to\infty} H_i(B\mathfrak{S}_k)$ is isomorphic to $H_i(B\mathfrak{S}_k)$ for k large enough. We call these groups the *stable homology groups*. The stable homology is computable in homotopy-theoretic terms, as was done by Barratt, Quillen, Priddy and Segal [Seg73].

Theorem 2.3. *We have an isomorphism*

$$\mathrm{colim}_{k\to\infty} H_*(B\mathfrak{S}_k) \cong H_*(\Omega^\infty_0 S^\infty)$$

where $\Omega^\infty_0 S^\infty$ *is a component of* $\mathrm{colim}_{n\to\infty}\Omega^n S^n$ *and* Ω *is the based loop space.*

There is in fact a geometric interpretation for this isomorphism, which we will see later. The right hand side of this theorem may seem abstract, but it is actually computable using iterated Eilenberg-Steenrod spectral sequences. Rationally the stable homology is zero, but this is unsurprising: each $\mathfrak{S}_k$ is a finite group and hence has no rational homology (in positive degree). However, there is a lot of homology with finite field coefficients [CLM76]:

Theorem 2.4 (Cohen-May-Lauda)**.** *We have that* $H_*(\Omega^\infty_0 S^\infty; \mathbb{F}_2)$ *is the free graded commutative algebra on* $Q^I x$ *with* I *ranging over admissible sequences,* Q^i *an Araki-Kudo-Dyer-Lashof operation and* $x \in H_0$. *There is a similar answer for* $\mathbb{F}_p$*-coefficients, which additionally involves the mod* p *Bockstein.*

Remark that $B\mathfrak{S}_k$ classifies manifold bundles with fibers consisting of sets of k points, that is, k-fold covering spaces. Thus we have found a large collection of characteristic classes for covers. They have no known geometric interpretation.

3. Not just some, but all examples

There are many more examples: see table on the last page. The story is roughly that even-dimensional manifolds of dimension not 4 are understood pretty well. We have stability for many odd-dimensional cases but no idea for the stable homology, and the stable homology in dimension 4 but no idea for stability.

Example 3.1. Let's go through one of the classical examples, oriented surfaces, so the reader has an idea how to read the table. In this case we consider $\mathrm{Diff}(\#_g(S^1 \times S^1), D^2)$, the diffeomorphisms of a genus g surface fixing a disk. This is isomorphic to $\mathrm{Diff}(\#_g(S^1 \times S^1)\backslash\mathrm{int}(D^2), \partial D^2)$, the diffeomorphisms of a genus g surface with one boundary component that fix the boundary. Take a torus with two disks removed. We can glue this to $\#_g(S^1 \times S^1)\backslash\mathrm{int}(D^2)$ along the boundary, thus increasing the genus by 1. This induces a map t on diffeomorphism groups by extending a diffeomorphism by the identity on the torus. The table then contains the Harer-Ivanov-Boldsen theorem [Wah11] that

$$t_* : H_*(B\mathrm{Diff}(\#_g(S^1 \times S^1)\backslash\mathrm{int}(D^2), \partial D^2)) \to H_*(\mathrm{Diff}(\#_{g+1}(S^1 \times S^1)\backslash\mathrm{int}(D^2), \partial D^2))$$

is an isomorphism for in a range increasing to infinity as $g \to \infty$ (in fact, the range is $* < \frac{2}{3}g$, which is nearly known to be optimal) and the Madsen-Weiss theorem [GMTW10] that the stable homology is that of an component of $\Omega^{\infty-1}MTSO(2)$. Here $MTSO(2)$ is the Thom spectrum of $-\gamma$ over $BSO(2)$, where γ is the universal bundle. Homology of Thom spectra is easy to understand if you're into that type of thing.

4. 0-dimensional manifolds

We will now prove homological stability and stable homology for symmetric groups.

Remark 4.1. Exactly the same proof works for $\mathrm{Diff}(\sqcup_k M)$ of a connected M. The range will be the same, the stable homology will be $H_*(\Omega^\infty_0 \Sigma^\infty B\mathrm{Diff}_+)$.

4.1. A model for $B\mathfrak{S}_k$.

We start by describing a model for $B\mathfrak{S}_k$, which is really one is a weak homotopy type determined by being connected and satisfying $\Omega B\mathfrak{S}_k \simeq \mathfrak{S}_k$. There are many choices of models for this weak homotopy type, and we opt for a geometric one. The important property of BG is the existence of a universal bundle over it, as described when we were talking about characteristic classes. It is easy to recognize universal bundles: principal G-bundles over a space B is universal if the total space E is contractible. This determines B up to weak equivalence and gives an concrete method to describe B if G is finite: if one finds a contractible space E with properly discontinuous free G-action, then E/G is a model for BG.

We will find such a space E for $G = \mathfrak{S}_k$. In the process we will also find useful finite-dimensional approximations to BG. To construct E we need something on which $\mathfrak{S}_k$ clearly acts properly discontinuously and freely: one such example is the space $\mathrm{Emb}([k], \mathbb{R}^N)$ of embeddings of the set $[k] := \{1, \ldots, k\}$ into $\mathbb{R}^N$.

Lemma 4.2. *We have that* $\mathrm{Emb}([k], \mathbb{R}^N)$ *is* $(N/2 - 2)$*-connected.*

Proof. This is a consequence of the qualitative Whitney embedding theorem, but we will give an elementary proof. To prove that $\mathrm{Emb}([k], \mathbb{R}^N)$ is $(N-1)$-connected, we need to extend any map $S^i \to \mathrm{Emb}([k], \mathbb{R}^N)$ to a map $D^{i+1} \to \mathrm{Emb}([k], \mathbb{R}^N)$ for any $i \leq N-1$. A map $S^i \to \mathrm{Emb}([k], \mathbb{R}^N)$ is the same as a smooth map $S^i \times [k] \to \mathbb{R}^N$ that is injective when restricted to each $x \in S^i$. We can always extend this to a smooth map $D^{i+1} \times [k] \to \mathbb{R}^N$, but we want to be injective for all $y \in D^{i+1}$. One easy way to arrange this is when the map itself is injective. But a generic map $D^{i+1} \times [k] \to \mathbb{R}^N$ has transverse self-intersections of dimensions$2(i+1) - N$, which is empty if $i < N/2 - 1$. $\square$

There are maps $\mathrm{Emb}([k], \mathbb{R}^N) \to \mathrm{Emb}([k], \mathbb{R}^{N+1})$ by including $\mathbb{R}^N$ into $\mathbb{R}^{N+1}$. The previous lemma implies that

$$\mathrm{Emb}([k], \mathbb{R}^\infty) := \mathrm{colim}_{N\to\infty}\mathrm{Emb}([k], \mathbb{R}^N)$$

is contractible and the map $\mathrm{Emb}([k], \mathbb{R}^N) \to \mathrm{Emb}([k], \mathbb{R}^\infty)$ is highly-connected. This means that $\mathrm{Emb}([k], \mathbb{R}^\infty)/\mathfrak{S}_k$ is a model for $B\mathfrak{S}_k$ and $\mathrm{Emb}([k], \mathbb{R}^N)/\mathfrak{S}_k$ is an approximation to it. The conclusion is that $B\mathfrak{S}_k$ should intuitively be thought as a space of k unlabeled distinct particles in a high-dimensional Euclidean space.

4.2. **Homological stability.** For homological stability it will useful to write $\mathbb{R}^N$ as $(0,\infty) \times \mathbb{R}^{N-1}$. We can easily write the stabilization map $t : B\mathfrak{S}_k \to B\mathfrak{S}_{k+1}$ in this model. This map needs to add one new particle, which is done by adding it near the origin, making place for it by shifting the first coordinates of all the other particles:

$$t(\vec{x}) = (\vec{x} + \vec{e}_1) \cup \text{new particle at } \frac{1}{2}\vec{e}_1$$

Recall our vague inspirations words for homological stability. The stabilization map t adds a new particle, and it would be easy to prove stability if this map had an inverse, which one imagines looks like removing a particle. Of course such a map doesn't exist, as there isn't a canonical choice for which of the $(k+1)$ particles to remove. In this case we do what topologists always do: make a space of choices. A first approximation would be the space X_0, consisting of a point $\vec{x} \in B\mathfrak{S}_{k+1} = \mathrm{Emb}([k+1], (0,\infty) \times \mathbb{R}^\infty)/\mathfrak{S}_k$ and a path γ_0 from the origin to one of the elements of $\vec{x}$ avoiding all other elements of $\vec{x}$. This space is close to what we want: $X_0 \simeq B\mathfrak{S}_k$ by dragging the point to the origin along γ and under this homotopy equivalence the map $X_0 \to B\mathfrak{S}_{k+1}$ which forgets the path is homotopic to t. However, the problem is that for a given $\vec{x} \in B\mathfrak{S}_{k+1}$ we have essentially constructed $k+1$ inverses to the stabilization map. We want to show that these are the same on homology, and thus build more spaces allowing interpolations between them.

Definition 4.3. The augmented semisimpicial space $X_\bullet$ has X_p equal to $\vec{x} \in B\mathfrak{S}_{k+1}$ together with an ordered collection of $(p+1)$ paths from the origin to points in $\vec{x}$, which are disjoint except at the origin. There are maps $d_i : X_p \to X_{p-1}$ for $0 \leq i \leq p$ which forget the ith path, and there is a map $\epsilon : X_0 \to B\mathfrak{S}_{k+1}$ which forgets the remaining paths.

By the same argument as before $X_p \simeq B\mathfrak{S}_{k-p}$ and each of the d_i's is homotopic to t.

Let Δ^p be the subspace of $[0,1]^{p+1}$ consisting of t_i's whose sum is 1. There are inclusions $d^i : \Delta^{p-1} \to \Delta^p$ given by making the ith coordinate 0. Out of $X_\bullet$ we can built a space

$$||X_\bullet|| = \left(\bigsqcup_{k \geq 0} X_p \times \Delta^p \right) /\!\sim$$

where $\sim$ is generated by $(d_i x, t) \simeq (x, d^i t)$. Think of $||X_\bullet||$ as the space of $\vec{x} \in B\mathfrak{S}_{k+1}$ together with weighted collections of paths from the origin to points in $\vec{x}$. The weights sum to 1 and if the weight is 0 you forget that path.

Our goal is to show that $||X_\bullet||$ is a good approximation to $B\mathfrak{S}_k$. Let's consider the map $\epsilon : ||X_\bullet|| \to B\mathfrak{S}_{k+1}$. The fiber over $\vec{x}$ consists of all linear combinations of paths from the origin to $\vec{x}$.

Lemma 4.4. *The fibers of ϵ are $(k-1)$-connected.*

Proof. Let's for the moment forget about the paths and just remember the points in $\vec{x}$ that these points connect to. The result is that from a map $S^i \to \epsilon^{-1}(\vec{x})$ we get a map from S^i to the space of linear combinations of ordered subsets of $\vec{x}$. More precisely, there is a semisimplicial space $\mathrm{Inj}_\bullet(\vec{x})$ with $\mathrm{Inj}_p(k)$ ordered $(p+1)$-element subsets of $\vec{x} \cong \{0, \ldots, k\}$ and d_i forgetting the i the element of your subset. We get a map $S^i \to ||\mathrm{Inj}_\bullet(\vec{x})||$. It is well known that $||\mathrm{Inj}_\bullet(\vec{x})||$ is $(k-1)$-connected [RW13], which is plausible since forgetting about the ordering would lead to a simplicial complex homeomorphic to a simplex. This semisimplicial space being highly connected means that we can extend our choices of elements to D^{i+1} if $i \leq k-1$.

What about the paths? A map $S^i \to \epsilon^{-1}(\vec{x})$ is a map out of a compact space. This means that the paths in its image are contained in $(0,\infty) \times \mathbb{R}^N$ for some N. Thus by going into an additional dimension allows us to endow our choices of subsets for $\vec{x}$ for each point in D^{i+1} with disjoint paths connecting them to the origin. This is formalized by the so-called lifting argument for connectivity of complexes [HW10, GRW14b, Kup13]. $\square$

So we have a map $\epsilon : ||X_\bullet|| \to B\mathfrak{S}_{k+1}$ with highly-connected fibers. But it is in fact a Serre fibration: if we have a map $D^i \times [0,1] \to B\mathfrak{S}_{k+1}$, i.e. the points in $\mathfrak{S}_{k+1}$ move around, by the isotopy extension theorem there exist a $D^i \times [0,1]$-indexed family compactly-supported isotopies of the ambient space starting at the identity inducing this. Now apply these isotopies to the paths to get a lift to $||X_\bullet||$. Thus the map $||X_\bullet|| \to B\mathfrak{S}_{k+1}$ is in fact $(k-1)$-connected and $||X_\bullet||$ looks like $B\mathfrak{S}_{k+1}$ in a range, in particular has the same homology.

To get information about the homology of $||X_\bullet||$ we filter it by

$$||X_\bullet||_{\leq j} = \left(\bigsqcup_{j \leq k \geq 0} X_p \times \Delta^p \right) /\sim$$

so that the associated gradeds are suspension of X_p's. The result is a spectral sequence with E^1-page given by

$$E^1_{p,q} = H_q(B\mathfrak{S}_{k-p})$$

and converging to $H_{p+q}(B\mathfrak{S}_{k+1})$ in a range. The d_1-differential is given by the alternating sum of the face maps. But we saw that all of them are homotopic to t, so they are alternatively t and 0. Furthermore, the edge map from $E^1_{0,q} = H_q(B\mathfrak{S}_k) \to H_q(B\mathfrak{S}_{k+1})$ is also given by t_*.

The proof now goes by induction over k and clearly the stabilization map is an isomorphism on H_0. So suppose have proven the result for all $k' \leq k$ and we want to prove it for $k+1$. Applying the inductive hypothesis when passing to the E^2-page results in the $p = 0$ column surviving, but the other columns being 0 in a triangular range. This means that the edge map, i.e the stabilization map $H_q(B\mathfrak{S}_k) \to H_q(B\mathfrak{S}_{k+1})$ is an isomorphism in a range, since nothing else can contribute to E^∞.

4.3. **Scanning.** How does one obtain the stable homology? For that we use the approximating spaces $\mathrm{Emb}([p], \mathbb{R}^N)/\mathfrak{S}_p$. We will construct a map

$$\mathbb{R}^N \times \mathrm{Emb}([p], \mathbb{R}^N)/\mathfrak{S}_p \to S^N = \mathbb{R}^\infty \cup \{\infty\}$$

and show it extends to a map $S^N \wedge (\mathrm{Emb}([p], \mathbb{R}^N)/\mathfrak{S}_p)_+ \to S^N$, hence map $\mathrm{Emb}([p], \mathbb{R}^N)/\mathfrak{S}_p \to \Omega^N S^N$. These maps are compatible for different N and taking N to infinity gives the desired *scanning map* $B\mathfrak{S}_k \to \Omega^\infty S^\infty := \mathrm{colim}_{N\to\infty} \Omega^N S^N$ [McD75] [Seg73]. Note that since $B\mathfrak{S}_k$ is connected, it hits a single component of the right hand side. Since all components of the right hand side, indexed by $\mathbb{Z}$, are homotopy equivalent, without loss of generality we can take it to be the zero component.

The idea is for $(y, \vec{x})$ take a tiny microscope centered around y and record what you see. Tiny here means that the diameter of the disk you see in your microscope is smaller than the distance between the particles in $\vec{x}$, so you see at most 1 point. Thus the assignment $(y, \vec{x}) \mapsto \tilde{s}(y, \vec{x}) \in \mathbb{R}^N \cup \{\infty\}$ is given by recording either the position of the particle in your vision (rescaled to be $\mathbb{R}^N$) or nothing, in which you send it to ∞. It is easy to see that the size of the microscope can be chosen continuously in $\vec{x}$, so we get a continuous map. Also note that if you go sufficiently far from the origin, you will always see the empty configuration in your microscope, which means we can extend our map to a map $S^N \wedge (\mathrm{Emb}([p], \mathbb{R}^N)/\mathfrak{S}_p)_+ \to S^N$.

We claim that this map induces an isomorphism on stable homology, that is, we get an isomorphism

$$s_* : \mathrm{colim}_{k\to\infty} H_*(B\mathfrak{S}_k) \to H_*(\Omega^\infty_0 S^\infty)$$

The argument for this is a combination of the group completion theorem and a delooping argument. The former says [MS76].

Theorem 4.5 (McDuff-Segal)**.** *If M is a homotopy commutative topological monoid, then*

$$H_*(M)[\pi_0^{-1}] \cong H_*(\Omega BM)$$

In our case $M_N = \bigsqcup_p \mathrm{Emb}([p], \mathbb{R}^N)/\mathfrak{S}_p$. This is a homotopy commutative topological monoid by juxtaposition: $\mathbb{R}^N \cong \mathbb{R}^{N-1} \times (0,1)$ and we can put two configurations in $\mathbb{R}^{N-1} \times (0,1)$ next to each other in $\mathbb{R}^{N-1} \times (0, 2)$ and rescale the last coordinate. There is an isomorphism for $i > 0$:

$$H_i \left(\bigsqcup_p \mathrm{Emb}([p], \mathbb{R}^N)/\mathfrak{S}_p \right) [\pi_0^{-1}] \cong \mathrm{colim}_{k\to\infty} H_i(B\mathfrak{S}_k)$$

Thus we want to compute the homology of ΩBM_N, because in a range it coincides with that of ΩBM_∞. But it turns out that BM_N is modeled by the space of particles in $\mathbb{R}^N$ that go and disappear at infinity in one direction. This is path-connected and again a topological monoid by juxtaposition in one of the remaining directions.

Theorem 4.6. *If M is a path-connected topological monoid, then*

$$M \simeq \Omega BM$$

Applying this theorem $(N-1)$ times, we see then ΩBM_N is homotopy equivalent to $\Omega^N B^N M_N$. We have that $B^N M_N$ is modeled by particles in $\mathbb{R}^N$ that can go to infinity in all N directions. Putting one particle at $y \in \mathbb{R}^N$ gives a map $\mathbb{R}^N \to B^N M_N$, which extends to a map $S^N \to B^N M_N$ by sending the new point to the empty configuration. This map is a weak equivalence; just zoom in until you see at most one point. We conclude that in the range $i \leq N/2 - 2$ we have

$$\mathrm{colim}_{k\to\infty} H_i(B\mathfrak{S}_k) = H_i(\Omega^N S^N)$$

and it remains to check that the isomorphism is induced by the scanning map described above. By homological stability the result doesn't depend on N in that range, so that we can take N to ∞. However, one can also draw a different conclusion.

Theorem 4.7 (Weaker version of Freudenthal suspension)**.** *If $N \geq 1$, the map $\Omega^N S^N \to \Omega^{N+1} S^{N+1}$ is $(N/2-2)$-connected.*

Proof. An H-space map that is a homology equivalence is a weak equivalence and a similar result holds in a range. □

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	Definition	Stability?	Stable homology?
Symmetric groups	$\mathfrak{S}_k \simeq \mathrm{Diff}(\sqcup_k *)$	Nakaoke [Nak60]	Barratt-Quillen-Priddy-Segal [Seg73], $\Omega^\infty \mathbb{S}$
Braid groups	$\mathrm{Br}_k \simeq \mathrm{Diff}(D^2 \backslash k \text{ points}; \partial D^2)$	Arnol'd [Arn70]	Segal [Seg73], $\Omega^2 S^2$
Punctured manifolds	$\mathrm{Diff}(M, k \text{ points})$	Segal-McDuff [Seg79] [McD75], Hatcher-Wahl [HW10]	McDuff [McD75], $\Gamma(\dot{T}M)//\mathrm{Diff}(M)$
Mapping class groups, or diffeomorphisms of surfaces	$\Gamma_{g,1} \simeq \mathrm{Diff}(\#_g(S^1 \times S^1), D^2)$, Γ_g	Harer, Ivanov, Boldsen [Har85]	Madsen-Weiss [MW07], $\Omega^\infty MTSO(2)$
Diffeomorphisms of non-orientable surfaces	$\mathrm{Diff}(N\#_g(S^1 \times S^1); D^2)$	Wahl [Wah08]	Galatius-Madsen-Tillman-Weiss [GMTW10], $\Omega^\infty MTO(2)$
Discrete diffeomorphisms of surfaces	$\mathrm{Diff}^\delta(\#_g(S^1 \times S^1), D^2)$	Nariman [Nar15]	Nariman [Nar15], $\Omega^\infty MTBS\Gamma_2$
Discrete symplectomorphisms of surfaces	$\mathrm{Symp}^\delta(\#_g(S^1 \times S^1), D^2)$	Nariman [Nar15]	Nariman [Nar15]
Connected sums of $S^1 \times S^2$	$\mathrm{Diff}(\#_g(S^1 \times S^2), D^3)$	Chor Hang Lam [HL15]	Hatcher [Hat12], $\Omega^\infty BSO(4)_+$
Diffeomorphisms of highly-connected manifolds, $\dim = 2n$, $n \geq 3$	$\mathrm{Diff}(\#_g(S^n \times S^n), D^{2n})$, $\mathrm{Diff}(\#_g(S^k \times S^{2n-k}), D^{2n})$	Galatius-Randal-Williams [GRW14b], Perlmutter [Per14]	Galatius-Randal-Williams [GRW12] $(\Omega^\infty MT\theta_{2n})//\mathrm{haut}(\theta_{2n})$
Discrete diffeomorphisms of highly-connected manifolds, $\dim = 2n$, $n \geq 3$	$\mathrm{Diff}^\delta(\#_g(S^n \times S^n), D^{2n})$	Nariman [Nar14]	Nariman [Nar14]
Block diffeomorphisms of even dimension $2n$, $n \geq 3$, rationally	$\widetilde{\mathrm{Diff}}(\#_g(S^n \times S^n); D^{2n})$	Berglund-Madsen (over $\mathbb{Q}$) [BM14]	Berglund-Madsen [BM14], explicit formula
Homotopy automorphisms of even dimension $2n$, $n \geq 3$, rationally	$\mathrm{haut}(\#_g(S^n \times S^n); D^{2n})$	Berglund-Madsen (over $\mathbb{Q}$) [BM14]	Berglund-Madsen [BM14], explicit formula

	Definition	Stability?	Stable homology?
Mapping class groups of 3-manifolds	$\pi_0\mathrm{Diff}(M\#_k P)$	Hatcher-Wahl [HW10]	
Diffeomorphisms of highly-connected manifolds, $\dim = 4n+1$, $n \geq 2$	$\mathrm{Diff}(W\#_g M, D^{4n+1})$	Perlmutter [Per13]	
Mapping class groups of 4-manifolds	$\pi_0\mathrm{Diff}(\#_g(S^2 \times S^2))$		Giansiracusa [Gia08], $BO_{\infty,\infty}(\mathbb{Z})^+$
Diffeomorphisms of smooth 4-manifolds	$\mathrm{Diff}(\#_g(S^2 \times S^2))$		Galatius-Randal-Williams [GRW12] $(\Omega^\infty MT\theta_4)//\mathrm{haut}(\theta_4)$
PL-Homeomorphisms of highly-connected manifolds, $\dim = 2n$, $n \geq 3$	$\mathrm{PL}(\#_g(S^n \times S^n), D^{2n})$	(working on it)	Gomez-Lopez [GL15], $\Omega^\infty \Phi_{\mathrm{PL}}$
Homeomorphisms of highly-connected manifolds, $\dim = 2n$	$\mathrm{Homeo}(\#_g(S^n \times S^n), D^{2n})$	(working on it)	(working on it)
Homotopy automorphisms of even dimension $2n$	$\mathrm{haut}(\#_g(S^n \times S^n))$		
Connected sums of $S^1 \times S^n$, $n \geq 3$	$\mathrm{Diff}(\#_g(S^1 \times S^n); D^{n+1})$		
Mapping class groups of highly-connected manifolds, $\dim = 2n$, $n \geq 3$	$\pi_0\mathrm{Diff}(W\#_g(S^n \times S^n))$		