UNDERSTANDING KISTER'S THEOREM

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Abstract. We give an exposition of Kister's proof that $\mathrm{Emb}^{TOP}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathrm{Top}(n)$, analogous to the more well-known result that $\mathrm{Emb}^{DIFF}(\mathbb{R}^n, \mathbb{R}^n) \simeq O(n)$. Finally, we explain how this prove that every topological microbundle contains a unique fiber bundle.

1. Introduction

Here are two of the most basic objects in manifold theory:

(i) The groups

$$\{\text{CAT-isomorphisms } (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\}$$

with CAT equal to $DIFF$, TOP or PL. In words, these are the diffeomorphisms, homeomorphisms and PL-homeomorphisms of $\mathbb{R}^n$ fixing the origin.

They can be topologized in the C^∞-topology and compact-open topology for TOP and $DIFF$ respectively, and need to be defined simplicially for PL, as there is no good topology on piecewise-linear maps. These topological groups are denoted $\mathrm{Diff}(n)$, $\mathrm{Top}(n)$ and $\mathrm{PL}(n)$ respectively.

(ii) The monoids

$$\{\text{CAT-embeddings } (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\}$$

with CAT equal to $DIFF$, TOP or PL. In words, these are the self-embeddings of $\mathbb{R}^n$ fixing the origin in different categories. They can topologized or defined simplically as in (i). These is no special notation for them, so we use $\mathrm{Emb}^{CAT}((\mathbb{R}^n, 0), (\mathbb{R}^n, 0))$.

Remark 1.1. Note that both of these include into the CAT-isomorphisms or CAT-embeddings that do not necessarily fix the origin. A translation homotopy, i.e. we deform the embedding ϕ by the family $\Phi(-, t) = \phi(-) - t\phi(0)$, shows that these inclusions are homotopy equivalences.

The reason there is no special notation for the self-embeddings is the following theorem.

Theorem 1.2. *The inclusion*

$$\{\textit{CAT-isomorphisms } (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\} \hookrightarrow \{\textit{CAT-embeddings } (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\}$$

is a weak equivalence if CAT is $DIFF$, TOP or PL.

We will in this note explain the cases $DIFF$ and TOP. The former will serve as an explanation of the proof strategy for the latter, but in a familiar setting. We slightly deviate from Kister's proof in [Kis64], making our deformations less canonical in return for a small gain in simplicity.

Why do we care about these results? Firstly, they are used in the theory of factorization homology. For say topological manifolds, this theory involves algebras over the operad with k-ary operations given by $\mathrm{Emb}^{TOP}(\bigsqcup_k \mathbb{R}^n, \mathbb{R}^n)$ and the theorem implies this is weakly equivalent to $F_k(\mathbb{R}^n) \times \mathrm{Top}(n)^k$.

Secondly, this proof is an easy case of the techniques used in Edwards-Kirby, which proves the local contractibility of spaces of homeomorphisms and topological embeddings. From this they deduce the topological isotopy theorem, an important ingredient in the theory of topological manifolds.

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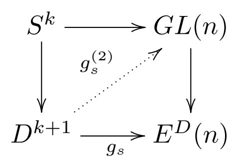
2. Smooth self-embeddings of $\mathbb{R}^n$

We start by proving that the all of the inclusions

$$O(n) \hookrightarrow GL(n) \hookrightarrow \mathrm{Diff}(n) \hookrightarrow \mathrm{Emb}^{DIFF}((\mathbb{R}^n,0),(\mathbb{R}^n,0))$$

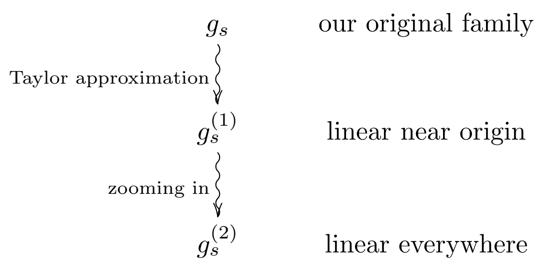
are weak equivalences. For convenience, we will denote $\mathrm{Emb}^{DIFF}((\mathbb{R}^n,0),(\mathbb{R}^n,0))$ by $E^D(n)$. Our strategy will be to prove this for $GL(n) \hookrightarrow E^D(n)$ and note that our proof will restrict to a proof for $GL(n) \hookrightarrow \mathrm{Diff}(n)$. This is easy, as a diffeomorphism is just a surjective smooth embedding by the inverse function theorem. That $O(n) \hookrightarrow GL(n)$ is a weak equivalence is well-known and a consequence of Gram-Schmidt orthonormalization. By the 2-out-of-3 property for weak equivalences, we finally get that all inclusions are weak equivalences.

A weak equivalence is a map that induces an isomorphism on all homotopy groups with all base points. We leave it to the reader to check that this is equivalent to the existence of dotted lifts for each $k \geq -1$ in each diagram



making both triangles commute up to homotopy. That is, we want to deform the family g_s, $s \in D^{k+1}$ to linear maps, staying in linear maps if we are already in linear maps.

Our strategy is outlined by the following diagram:



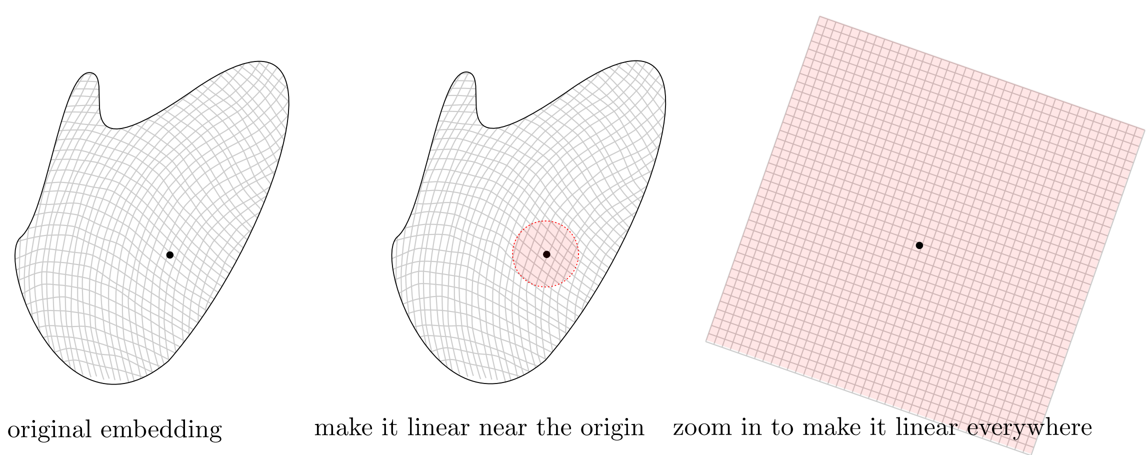


Figure 1. Our strategy in pictures.

(i) Our first step is making g_s linear near origin. Taylor approximation says that

$$||Dg_s(x) - Dg_s(0)|| \leq C||x||$$

for $||x|| \leq 1$ By compactness of D^{k+1} we can take C uniformly in s. So let $\eta : [0, \infty) \to [0, 1]$ be a decreasing function that is 1 near 0 and 0 on $[1, \infty)$ (it looks like a bump near the origin). Let's look at the function

$$G_s^{(1)}(x,t) = (1 - t\eta(||x||/\epsilon))g_s(x) + t\eta(||x||/\epsilon)Dg_s(0) \cdot x$$

This is a candidate for the isotopy deforming g_s to be linear near the origin, but we still need to pick ϵ. We have that $G_s(x,t) = g_s(x)$ for all $t \in [0,1]$ and $||x|| \geq \epsilon$. Let's consider for $||x|| \leq \epsilon$

$$||DG_s^{(1)}(x,t) - Dg_s(x)|| \leq ||t\eta(||x||/\epsilon)Dg_s(0) - t\eta(||x||/\epsilon)Dg_s(x)|| + ||tD\eta(||x||/\epsilon)g_s(x)||$$

The first term can be estimated as $\leq 2C\epsilon$, while the second term can be estimated by $\leq D\epsilon$ for D uniform in s. So picking ϵ small enough we can guarantee that $DG_s(x,t) \neq 0$ for all $x \in \mathbb{R}^n$, $t \in [0,1]$. This implies that they are all embeddings, since they are for $t = 0$. Indeed, the number of points in an inverse image cannot change without a critical point appearing. We set

$$g_s^{(1)} = G_s^{(1)}(-,1)$$

and note that if g_s was linear, then all $G_s^{(1)}(-,t)$ (and in particular $g_s^{(1)}$) are as well.

(ii) Our second step zooms in on the origin. To do this, we define

$$G_s^{(2)}(x,t) = \begin{cases} Dg_s^{(1)}(0) \cdot x & \text{if } ||x|| \leq \epsilon \text{ and } t < 1, \text{ or if } t = 1 \\ g_s^{(1)}(\frac{x}{1-t}) & \text{if } ||x|| > \epsilon \text{ and } t < 1 \end{cases}$$

We set $g_s^{(2)} = G_s^{(2)}(-,1)$ and note if $g_s^{(1)}$ was linear, then all $G_s^{(2)}(-,t)$ (and in particular $g_s^{(2)}$) are as well

This completes the proof that the map $GL(n) \hookrightarrow E^D(n)$ is a weak equivalence. We remark that if all g_s were diffeomorphisms, i.e. surjective, then so are all $G_s^{(1)}(-,t)$ and $G_s^{(2)}(-,t)$ and thus the same argument tells us that $GL(n) \hookrightarrow \mathrm{Diff}(n)$ is a weak equivalence.

3. Topological self-embeddings of $\mathbb{R}^n$

We next repeat the entire exercise in the topological setting. We want to show that the inclusion

$$\mathrm{Top}(n) \hookrightarrow \mathrm{Emb}^{TOP}((\mathbb{R},0),(\mathbb{R},0))$$

is a weak equivalence. For convenience, we will denote $\mathrm{Emb}^{TOP}((\mathbb{R},0),(\mathbb{R},0))$ by $E^T(n)$. As before it suffices to find lifts

$$\begin{array}{ccc} S^k & \longrightarrow & \mathrm{Top}(n) \\ \downarrow & \overset{g_s^{(2)}}{\nearrow} & \downarrow \\ D^{k+1} & \xrightarrow[g_s]{} & E^T(n) \end{array}$$

making both triangles commute up to homotopy. That is, we want to deform the family g_s, $s \in D^{k+1}$ to homeomorphisms staying in homeomorphisms if we already are in homeomorphisms. Here it is helpful to remark that since a topological embedding is a homeomorphism onto its image, it is a homeomorphism if and only if it is surjective.

Our strategy is outlined by the following diagram:

g_s — our original family

↓ canonical circle jiggling

$g_s^{(1)}$ — image equals an open disk

↓ zooming in

$g_s^{(2)}$ — surjective

The important technical tool replacing Taylor approximation is the following "circle jiggling" trick. Let D_r denote the closed disk of radius r around the origin.

Lemma 3.1. *Suppose we have $f, h \in E^T(n)$ with $h(\mathbb{R}^n) \subset f(\mathbb{R}^n)$ and $h(D_b) \subset f(D_c)$ for $0 < b, c < \infty$. Then there exists an isotopy ϕ_t of $\mathbb{R}^n$ such that*

(i) $\phi_0 = \mathrm{id}$,
(ii) $\phi_1(h(D_b)) \subset f(D_c)$,
(iii) ϕ_t *fixes* $\mathbb{R}^n \backslash f(D_d)$ *for* $d > c$ *and* $h(D_a)$ *for* $a < b$.

This is continuous in f, h and a, b, c, d.

Proof. Since $h(\mathbb{R}^n) \subset f(\mathbb{R}^n)$, it suffices to work in f-coordinates. Our isotopy will be compactly supported in these coordinates, so we can extend by the identity to the complement of $f(\mathbb{R}^n)$ in $\mathbb{R}^n$.

We will now define some subsets and invite the reader to look at Figure 2. Let b' be the radius of the largest disk contained in $h(D_b)$ (in f-coordinates, remember) and a' the radius of the largest disk contained in $h(D_a)$. Our first attempt for an isotopy is to make ϕ_t piecewise-linearly scale the radii between a and d such that c moves to b'. This satisfies (i), (ii) and fixes $\mathbb{R}^n \backslash f(D_d)$. It might not fix $h(D_a)$.

This can be solved by a trick: we conjugate with a homeomorphism, described in h-coordinates as follows: piecewise-linear radial scale between 0 and b moving a to a'', where a'' is the radius of the largest disk contained in $f(D_{a'})$. In words, we temporarily decrease the size of $h(D_a)$ to be contained in $f(D_{a'})$, do our previous isotopy, and restore $h(D_a)$ to its original shape.

The continuity of this construction depends on the continuity of b', a' and a'', which we leave to the reader as an exercise in the compact-open topology. □

We can now prove that $\mathrm{Top}(n) \hookrightarrow E^T(n)$ is a weak equivalence.

(i) Our first step involves making the images of g_s into a (possibly infinite) open disk. Let $R_s(r)$ be the piecewise linear function $[0, \infty) \to [0, \infty)$ sending $i \in \mathbb{N}_0$ to the radius of the largest disk contained in $g_s(D_i)$. Then we can construct an element of $E^T(n)$ given in radial coordinates by $h_s(r, \varphi) = (R_s(r), \varphi)$. This satisfies $h_s(\mathbb{R}^n) \subset g_s(\mathbb{R}^n)$, $h_s(D_i) \subset g_s(D_i)$ for all $i \in \mathbb{N}_0$ and has image an open disk. It is continuous in s.

Our goal is to deform h_s to have the same image as g_s in infinitely many steps. For $t \in [0, 1/2]$ we use the lemma to push $h_s(D_1)$ to contain $g_s(D_1)$ while fixing $g_s(D_2)$. For $t \in [1/2, 3/4]$ we use the lemma to push the resulting image of $h_s(D_2)$ to contain $g_s(D_2)$ while fixing $g_s(D_3)$ and the resulting image of $h_s(D_1)$, etc. These infinitely many steps converge to an embedding since on each compact only finitely many steps are not the identity. The result is a family $H_s(-, t)$ in $E^T(n)$ such that $H_s(-, 1)$ has the same image as g_s. It is continuous in s since h_s is. So step (i) does this:

$$G_s^{(1)}(x, t) = H_s(H_s(-, 1)^{-1} g_s(x), 1 - t)$$

For $t = 0$, this is simply $g_s(x)$. For $t = 1$, this is $H_s(-, 1)^{-1}(g_s(x))$, which we denote by $g_s^{(1)}$ and has the same image as $h_s(x)$, i.e. a possibly infinite open disk.

(ii) There is a piecewise-linear radial isotopy moving $h_s(x)$ to the identity. It is given by moving the values of R_s at each integer i to i. We set

$$G_s^{(2)}(x, t) = K_s(-, 1 - t)^{-1} g_s^{(1)}(x)$$

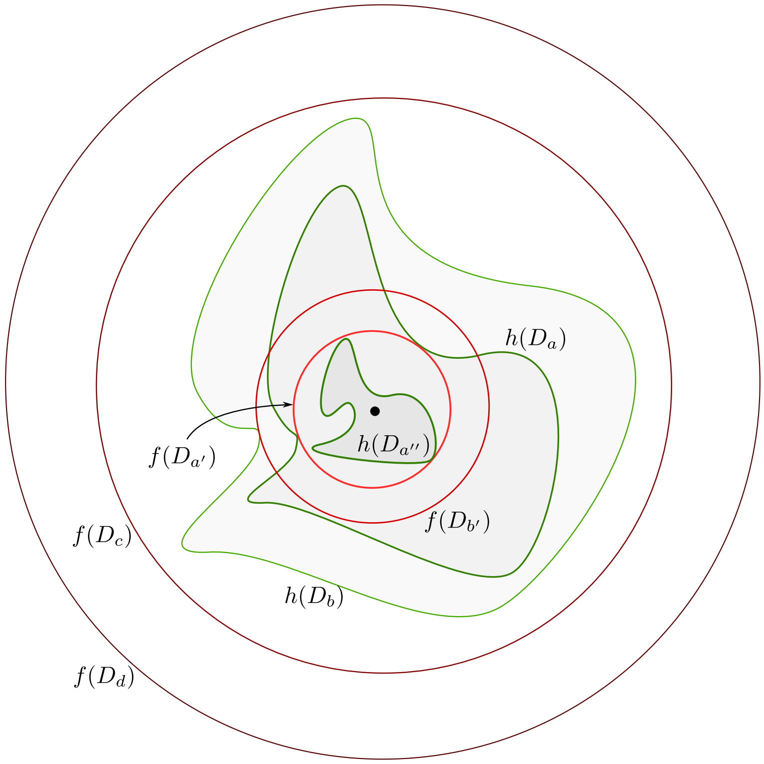


FIGURE 2. The various subsets of $\mathbb{R}^n$ appearing in Lemma 3.1 in f-coordinates (hence the $f(D_r)$ appear as actual disks). The $f(D_r)$ have red boundary, the $h(D_r)$ have green boundary.

so that for $t = 0$ we have get $g_s^{(1)}$ and for $t = 1$ we get $g_s^{(2)}$ with image $\mathbb{R}^n$.

4. Microbundles and fiber bundles

Microbundles are like vector bundles they exist more often [Mil64]. The reason for this is that instead of demanding local triviality on the entire fiber, we can only demand it near the zero-section. We finish this note with a sketch of the first application of the results in the previous section, the Kister-Mazur theorem saying that microbundles contain unique fiber bundles.

Definition 4.1. An n-dimensional microbundle over B is the data of $i : B \hookrightarrow E$ and $p : E \to B$ so that (i) $p \circ i = \mathrm{id}_B$ and (ii) for all $b \in B$ there exists an open V_b in E such that $p(V_b)$ is an open U_b in B and homeomorphisms making the following diagram commute

$$\begin{array}{ccccc} U_b & \xrightarrow{i} & V_b & \xrightarrow{p} & p(V_b) \\ \| & & \downarrow{\cong} & & \| \\ U_b \times \{0\} & \longrightarrow & U_b \times \mathbb{R}^n & \longrightarrow & U_b \end{array}$$

The basic example is that if M is a topological manifold of dimension n, then

$$TM = (M \overset{\Delta}{\hookrightarrow} M \times M \overset{\pi_2}{\to} M)$$

is the n-dimensional tangent microbundle.

Theorem 4.2 (Kister-Mazur)**.** *Every n-dimensional microbundle over a locally finite complex contains a fiber bundle with fiber $\mathbb{R}^n$ and transition functions in* $\mathrm{Top}(n)$, *unique up to isotopy.*

Sketch of proof. Let's prove existence. The proof is by reduction to finite complexes, and then induction over cells. Suppose we already have a bundle η over the $(k-1)$-skeleton and we try to extend it over a k-cell σ. By contractibility of the cell, the microbundle over it contains a trivial bundle, a fact proven by Milnor. By scaling in the fiber direction, we can in fact assume that this trivial bundle η_σ is arbitrarily small. In particular, over $\partial\sigma$ we can assume that this η_σ is contained in η to a collar. Using the two bundles we obtain fiberwise embeddings $g_s : (\eta_\sigma)_s \cong \mathbb{R}^n \to \mathbb{R}^n \cong \eta_s$ fixing the origin, where $s \in \partial\sigma$. By the result of the previous section is isotopic to a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ fixing the origin by $G_s(-,t)$. We can use this isotopy over a collar of $\partial\sigma$ to glue $\mathrm{im}(G_s(-,t))$ to η on one side and η_σ on the other side.

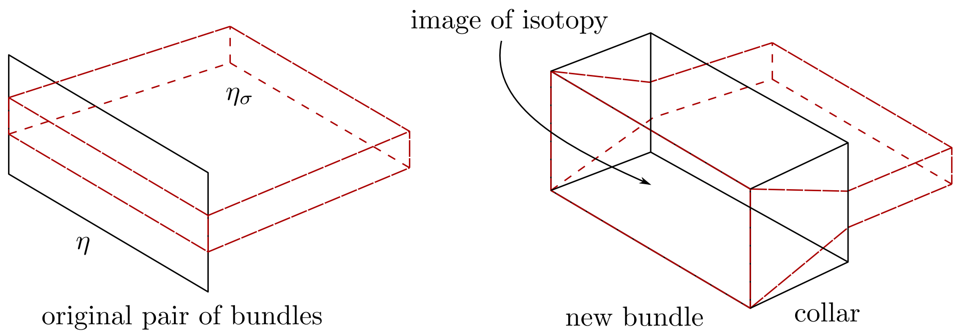


Figure 3. Extending the bundle.

To prove uniqueness note that bundles ξ_0 and ξ_1 contains a common bundle η. We get fiberwise embeddings $\eta \hookrightarrow \xi_i$ and can deform it to a homeomorphism cell by cell. This proves that ξ_0 is isotopic to η and hence by symmetry to ξ_1. □

References

[Kis64] J. M. Kister, *Microbundles are fibre bundles*, Ann. of Math. (2) **80** (1964), 190–199. MR 0180986 (31 #5216) 1
[Mil64] J. Milnor, *Microbundles. I*, Topology **3** (1964), no. suppl. 1, 53–80. MR 0161346 (28 #4553b) 5