Codimension 1 foliations on 3 manifolds : construction & classification

I. CONSTRUCTION

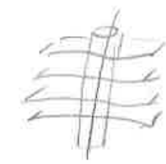
Thm (Lickorish, Novikov, Zieschang) 1960s ? Every closed orientable M^3 admits a smooth, transv. orient. codim 1 $\mathcal{F}$.

Proof outline. ① (Classical, A. Wallace) : Every M^3 can be obtained from S^3 by surgery (removing a finite collection of disjoint tori from S^3 & regluing)

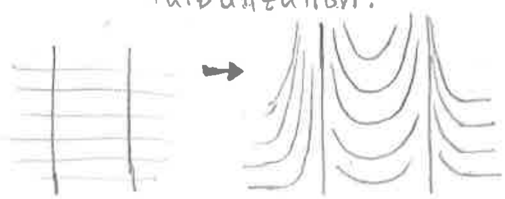
Example: $S^3 \setminus$ solid torus $D^2 \times S^1$ $= D^2 \times S^1$. reglue by id map, get $S^2 \times S^1$.

glued meridian to longitudes

② (After isotopy) can take core circles of surgery tori to be transverse to Reeb foliation on S^3



③ key construction : "Tourbillonnement" turbulization? Replace core-transverse (surgery) torus with Reeb compont.



cross-section

modification in nbd of torus

General version : S closed transversal to codim 1 $\mathcal{F}$ $\longrightarrow$ modify $\mathcal{F}$ on nbd of S to put Reeb component in nbhd of S.

Same regularity (smooth in coordinates from S)

④ Resulting foliation is C^r-trivial along boundary of torus (can just as well be so surgery torus & replacing is no problem.

II Classification ? "up to deformation"?

What is a good topology on $\mathcal{F}(M)$?

codim 1 transv. oriented foliatns

Have $\mathcal{F}(M) \hookrightarrow P(M)$, can take subspace topology.

closed subset (integrability)

2-plane fields (transv. orient.)

$\underline{Q}$: How closely are these related?

($P(M)$ is easier object to study ...)

<u>GOOD NEWS</u>: <u>Thm</u> (Wood, Thurston): Any plane field on M can be isotoped to an integrable one.

$\underline{Q}$: In families? If tangent plane fields to $\mathcal{F}_1$ & $\mathcal{F}_2$ are isotopic thru $P(M)$, can that path be taken to lie in $\mathcal{F}(M)$?

(Is $\pi_0 \mathcal{F}(M) \to \pi_0 P(M)$ injective?)

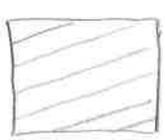
<u>A</u>: (Eynard-Bontemps, '15) Yes.

<u>WARNING</u> (and useful tool in proof!)

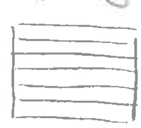
Isotopy of integrable plane fields $\neq$ "isotopy of leaves of $\mathcal{F}$"

<u>Ex 1</u>: Linear foliation of solid torus

cross-section:



varying Slope gives <u>smooth family of plane fields</u>

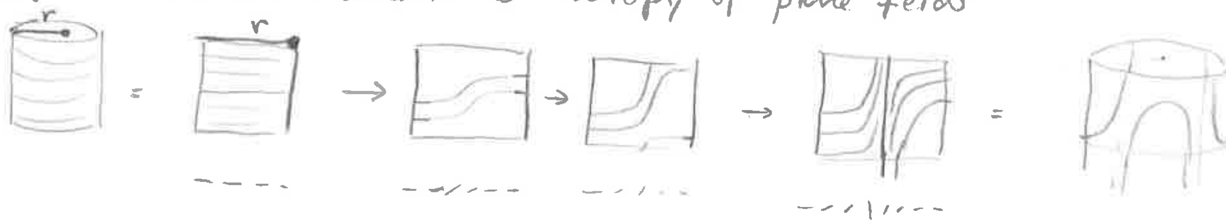


Slope 0: closed leaves

Slope $\varepsilon \notin \mathbb{Q}$: no closed leaves, all leaves dense.

boundary

Ex 2: Tourbionnement as isotopy of plane fields



VERY SHORT CARTOON ~~OUTLINE~~ OF PROOF of EYNARD-BONTEMPS' theorem:

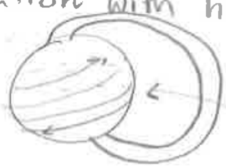
Follow strategy of Thurston's proof

... But modify to do in families.

- Find good triangulation of M
  "transverse to plane field", lets you isotope planes to be integrable near 2-skel.

→ Use techiques of Eliashberg (developed for plane fields of contact structures)

- Have foliation with holes.



3-cell

Reeb stability ⇒ can't typically fill the ball

OK. Not OK

So modify original foliation on solid torus, 2

fill here, transverse to S^1 direction.

(tourbionnement!)

→ do in families (Schweitzer, Larcanché)

"Foliated S^1- bundles" Extending over disc.

- Except transverse arcs don't always exist.. (problem w/ compact leaves)

  Fix this.

→ Argument to again reduce to having to eliminate torus leaves,

Key tool: connectedness of (holonomy)

Hom ($\mathbb{Z}^2$, Diff $[0,1]$)

$(\pi_1(T^2))$

COR. OF ARGUMENT:

easily show $\pi_k(\mathcal{F}(M)) \to \pi_k(P(M))$ is surjective for all k.