Connected components of representation spaces

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Representation spaces

$\Gamma =$ finitely generated group

$G =$ topological group

$\mathrm{Hom}(\Gamma, G) =$ space of homomorphisms $\Gamma \to G$.

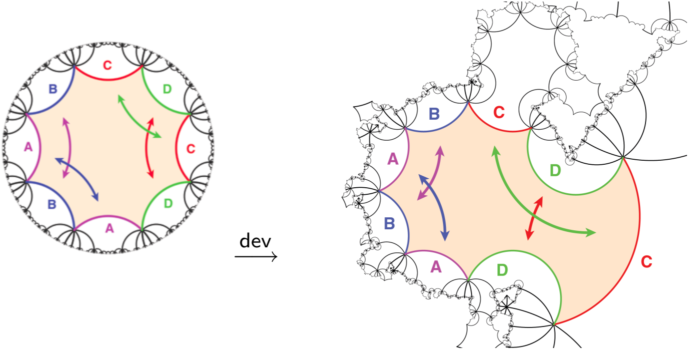
Natural topology as subset of $G^{|S|}$ $\quad$ S a generating set for Γ.

Hom(Γ, G): key interpretations

1. Geometric structures

M manifold, $\Gamma = \pi_1(M)$, $G \subset \text{Homeo}(X)$

A (G, X)-structure on M is determined by
$\rho \in \text{Hom}(\Gamma, G)$ (holonomy representation)
and ρ-equivariant developing map, $\widetilde{M} \to X$



[W. Goldman "What is a projective structure?"]

$\mathrm{Hom}(\Gamma, G)$: key interpretations

2. Space of Γ-actions

M manifold, $\Gamma = \pi_1(M)$, $G \subset \mathrm{Homeo}(X)$

$\mathrm{Hom}(\Gamma, G)$ = space of Γ-actions on X

G specifies regularity of action, e.g. $G = \mathrm{Isom}(X)$, $G = \mathrm{conf}(X)$,
$G = \mathrm{Diff}^r(X)$.

Regularity matters!
Example: $\mathrm{Hom}(\mathbb{Z}^n, \mathrm{Diff}^1_+(S^1))$
recently shown to be *connected* (A. Navas, 2013)
Is $\mathrm{Hom}(\mathbb{Z}^n, \mathrm{Diff}^\infty_+(S^1))$ connected? Locally connected? (open)

Hom(Γ, G): key interpretations

3. Hom(Γ, G) = space of flat G-bundles

M manifold, $\Gamma = \pi_1(M)$, $G \subset \mathrm{Homeo}(X)$

$$\left\{ \begin{array}{c} \text{flat } X\text{-bundles over } M \\ \text{with structure group } G \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{representations} \\ \pi_1(M) \to G \end{array} \right\} = \mathrm{Hom}(\Gamma, G)$$

bundle $\leftrightarrow$ monodromy representation

equivalent bundles $\leftrightarrow$ conjugate representations

Connected components

Connected components of $\mathrm{Hom}(\Gamma, G)$ correspond to
deformation classes of structures
actions
bundles

Basic question: classify components

Example: $G \subset \mathrm{GL}(n, \mathbb{C})$, Lie group

$\Rightarrow$ $\mathrm{Hom}(\Gamma, G)$ is an affine variety

$\Rightarrow$ finitely many components

Classical example: $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R}))$

$\Gamma_g := \pi_1(\Sigma_g)$, $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Homeo}_+(S^1)$

- Theorem (Goldman, 1980)
  Components of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R}))$ are completely distinguished by the Euler number, $\mathrm{e}(\rho)$.

- Milnor-Wood inequality (1958)
  $\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R})) \Rightarrow -(2g-2) \leq \mathrm{e}(\rho) \leq 2g-2$

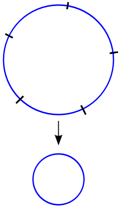
  $\Rightarrow \mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R}))$ has 4g-3 components

- Two components of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R}))$ are *Teichmüller space*
  $=$ space of hyperbolic structures on Σ_g.
  $=$ set of discrete, injective representations $\Gamma_g \to \mathrm{PSL}_2(\mathbb{R})$
  $=$ components where $\mathrm{e}(\rho)$ is maximal/minimal

- *Higher Teichmüller theory* studies $\mathrm{Hom}(\Gamma_g, G)$, G Lie group.

$\mathrm{Hom}(\Gamma_g, G)$, $G \subset \mathrm{Homeo}_+(S^1)$

Known: G a Lie group.

- $\mathrm{Hom}(\Gamma_g, S^1)$ is connected
- $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R}))$ has $4g-3$ components, distinguished by $\mathrm{e}(\rho)$
- $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}^{(k)})$ $\quad 1 \to \mathbb{Z}/k\mathbb{Z} \to \mathrm{PSL}^{(k)} \to \mathrm{PSL}_2(\mathbb{R}) \to 1$
- 

Theorem (Goldman, 1980)
Components of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}^{(k)})$ are distinguished by $\mathrm{e}(\rho)$, unless $k|(2g-2)$

If $k|(2g-2)$, there are k^{2g} components where $\mathrm{e}(\rho) = \pm\frac{2g-2}{k}$ (the maximal/minimal values).

Flat circle bundles over surfaces

$G = \mathrm{Homeo}_+(S^1)$

What are the connected components of $\mathrm{Hom}(\Gamma_g, G)$?

- more representations (even up to conjugacy)
- but easier to form paths between representations

Open Question

Does $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1))$ *have finitely many components?*
Does $\mathrm{Hom}(\Gamma_g, \mathrm{Diff}_+(S^1))$ *have finitely many components?*

Our results: Lower bound on number of components

Theorem 1 (M-)

For each divisor $k \neq \pm 1$ *of* $2g-2$,
There are at least $k^{2g}+1$ *components of* $Hom(\Gamma_g, \mathrm{Homeo}_+(S^1))$ *where* $\mathrm{e}(\rho) = \frac{2g-2}{k}$

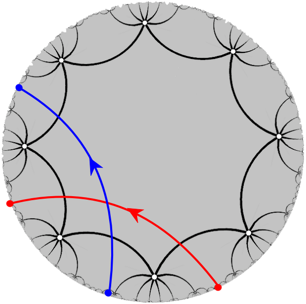
i.e. $\mathrm{e}(\rho)$ does not distinguish components

and $\mathrm{Hom}(\Gamma_4, \mathrm{Homeo}_+(S^1))$ has ≥ 165 components...

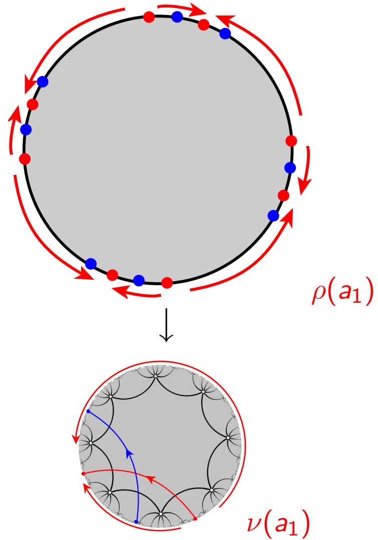
Moreover, two representations into $\mathrm{PSL}^{(k)}$ *that lie in different components of* $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}^{(k)})$ *cannot be connected by a path in* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1))$.

A picture: $\rho : \Gamma_g \to \mathrm{PSL}^k$ with $\mathrm{e}(\rho) = \frac{2g-2}{k}$

Start with $\nu : \Gamma_g \to \mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{e}(\nu) = 2g - 2$.



Lift to k-fold cover of S^1 for $\rho : \Gamma_g \to \mathrm{PSL}^k$ with $\mathrm{e}(\rho) = \frac{2g-2}{k}$


ρ(a1)
ν(a1)

Our results: Rigidity phenomena

Theorem 2 (M-)

Let $\rho : \Gamma_g \to \mathrm{PSL}^{(k)}$, $\mathrm{e}(\rho) = \pm(\frac{2g-2}{k})$. *Then*

Connected component of ρ *in* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1))$	$=$	*Semiconjugacy class of* ρ *in* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1))$

J. Bowden (2013): similar conclusion for $\mathrm{Hom}(\Gamma_g, \mathrm{Diff}_+^\infty(S^1))$, fundamentally different techniques. Uses smoothness.

Our key tool: rotation numbers

Theorem (*Rotation number rigidity*; M-)

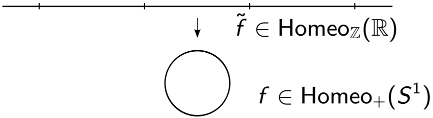
Let $\rho : \Gamma_g \to \mathrm{PSL}^{(k)}, \mathrm{e}(\rho) = \pm(\frac{2g-2}{k})$, $\gamma \in \Gamma_g$. *Then* $\mathrm{rot}(\rho(\gamma))$ *is constant under deformations of* ρ *in* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1))$.

$$\mathrm{rot}(\rho(\gamma)) = \mathrm{rot}(\rho_t(\gamma))$$

Rotation numbers

Definition (Poincaré)

rot : $\mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$. $\quad$ rot$(f) := \lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n} \mod \mathbb{Z}$



- continuous
- $\mathrm{rot}(f) = p/q \Rightarrow f$ has periodic point of period q
- $\mathrm{rot}(f^m) = m\,\mathrm{rot}(f)$

Similarly, define $\widetilde{\mathrm{rot}} : \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ by $\widetilde{\mathrm{rot}}(\tilde{f}) := \lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n} \in \mathbb{R}$

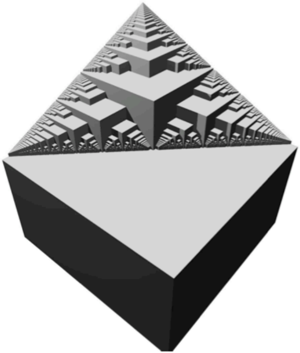
- Depends on lift $\tilde{f}$ (not just f).

However, a commutator $[f, g] \in \mathrm{Homeo}_+(S^1)$ has a *distinguished lift* $[\tilde{f}, \tilde{g}]$ so $\widetilde{\mathrm{rot}}[f, g]$ makes sense.

rot is not a homomorphism!

- It is possible to have $\widetilde{\text{rot}}(\tilde{f}) = \widetilde{\text{rot}}(\tilde{g}) = 0$ and $\widetilde{\text{rot}}(\tilde{f}\tilde{g}) = 1$

- Calegari-Walker (2011) give an algorithm to compute the *maximum value* of $\widetilde{\text{rot}}(\tilde{f}\tilde{g})$ given $\widetilde{\text{rot}}(\tilde{f})$ and $\widetilde{\text{rot}}(\tilde{g})$.



[D. Calegari, A. Walker, "Ziggurats and rotation numbers"]

- *But...* if $\tilde{f}\tilde{g} = T^n$ (translation by n), then $\widetilde{\text{rot}}(\tilde{f}) + \widetilde{\text{rot}}(\tilde{g}) = n$

Proof ideas for *rotation number rigidity* (Theorem 3)

Recall: Theorem 3
$\rho : \Gamma_g \to \mathrm{PSL}^{(k)}$, $\mathrm{e}(\rho) = \pm(\frac{2g-2}{k})$
$\Rightarrow$ $\mathrm{rot}(\rho(\gamma))$ constant under deformations of ρ.

Steps of proof:

1. The *Euler number* in terms of $\widetilde{\mathrm{rot}}$
2. Reduce to a question of *local maximality* of $\widetilde{\mathrm{rot}}(\tilde{f}\tilde{g})$
3. Dynamics and the Calegari-Walker algorithm

(4. Why $\widetilde{\mathrm{rot}}$ and $\mathrm{e}(\rho)$ are key.)

The Euler number $e(\rho)$

Classical definition is in terms of characteristic classes of circle bundles.

$$e_{\mathbb{Z}} \in H^2(\mathrm{Homeo}_+(S^1); \mathbb{Z}). \quad \langle \rho^*(e_{\mathbb{Z}}), [\Gamma_g] \rangle = e(\rho)$$

Definition (Milnor)

$\Gamma_g = \langle a_1, b_1, ... a_g, b_g \mid [a_1, b_1][a_2, b_2]...[a_g, b_g] \rangle$

$\rho : \Gamma_g \to \mathrm{Homeo}_+(S^1)$.

$$e(\rho) := \widetilde{\mathrm{rot}}\left([\tilde{\rho}(a_1), \tilde{\rho}(b_1)] \ldots [\tilde{\rho}(a_g), \tilde{\rho}(b_g)]\right)$$

- e is continuous on $\mathrm{Hom}(\Gamma_g, G)$ for any $G \subset \mathrm{Homeo}_+(S^1)$
- $[\rho(a_1), \rho(b_1)]...[\rho(a_g), \rho(b_g)] = \mathrm{id}$ on S^1
  $\Rightarrow [\tilde{\rho}(a_1), \tilde{\rho}(b_1)]...[\tilde{\rho}(a_g), \tilde{\rho}(b_g)] = T^{e(\rho)}$

Step 2. (A question of local maximality)

ρ_t path in $\mathrm{Hom}(\Gamma_2, \mathrm{Homeo}_+(S^1))$, ρ_0 as in Theorem.

$$[\tilde{\rho}_t(a_1), \tilde{\rho}_t(b_1)]\ [\tilde{\rho}_t(a_2), \tilde{\rho}_t(b_2)] = T^{e(\rho_t)}$$

$$\tilde{\mathrm{r}}\mathrm{ot}([\rho_t(a_1), \rho_t(b_1)]) + \tilde{\mathrm{r}}\mathrm{ot}([\rho_t(a_2), \rho_t(b_2)]) \equiv \mathrm{e}(\rho_0)$$

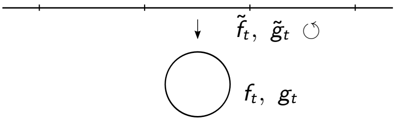
If we show: $\tilde{\mathrm{r}}\mathrm{ot}([\rho_t(a_i), \rho_t(b_i)])$ has local max at $t = 0$,
then we know $\tilde{\mathrm{r}}\mathrm{ot}([\rho_t(a_i), \rho_t(b_i)])$ is constant.

From here, same kind of work shows that that $\mathrm{rot}(\rho_t(a_i))$ and $\mathrm{rot}(\rho_t(b_i))$ are both constant, ...and also $\mathrm{rot}(\rho_t(\gamma))$ constant for any $\gamma \in \Gamma$.

Step 3. Dynamics and the Calegari-Walker algorithm

$f_0, \; g_0 \in \mathrm{Homeo}_+(S^1)$
$\tilde{f}_0, \; \tilde{g}_0 \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

f_t, g_t deformations. Lift to paths $\tilde{f}_t$, $\tilde{g}_t$ in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$

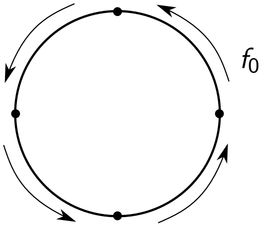


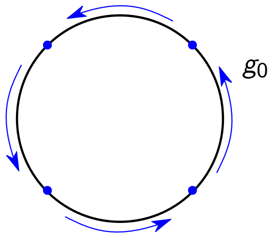
Study $t \mapsto \tilde{\mathrm{rot}}(\tilde{f}_t \circ \tilde{g}_t)$.

When is $t = 0$ a local maximum?

Step 3. Dynamics and the Calegari-Walker algorithm

Toy example:



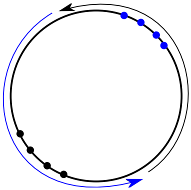


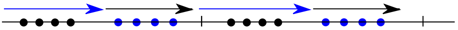
Claim: $\mathrm{rot}(f_0 \circ g_0) = 1/4$

$\tilde{\mathrm{rot}}(\tilde{f}_0 \circ \tilde{g}_0) = 1/4$

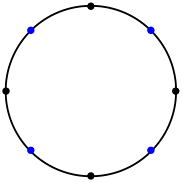
Step 3. Dynamics and the Calegari-Walker algorithm

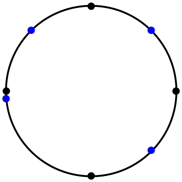
dynamics at global maximum:





at local maximum:





Why look at rot and e?

rot and e "essentially determine the dynamics of a representation".

Theorem (Ghys)

Γ any finitely generated group.
$\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$ is determined (up to semiconjugacy) by the bounded Euler class $\rho^*(e_{\mathbb{Z}}) \in H^2_b(\Gamma; \mathbb{Z})$.

Theorem (Matsumoto)

Γ any finitely generated group.
$\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$ is determined (up to semiconjugacy) by the bounded Euler class $\rho^*(e_{\mathbb{R}}) \in H^2_b(\Gamma; \mathbb{R})$ and the rotation numbers of a set of generators for Γ.

For $\rho : \Gamma \to \mathrm{Diff}^2_+(S^1)$, determined up to *conjugacy*

Ghys' and Matsumoto's theorems let us use *Rotation number rigidity* to prove Theorems 1 and 2.

Work in progress

- Do other representations satisfy rigidity properties?
- Distinguish other components of $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1))$.

  Example: is $\{\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1)) : \mathrm{e}(\rho) = 0\}$ connected?
- $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}_+(S^1))$ vs. $\mathrm{Hom}(\Gamma_g, \mathrm{Diff}_+(S^1))$
- What can analyzing rotation numbers tell us about components of $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))$, for other Γ? (e.g. 3-manifold groups)