Large scale geometry of homeomorphism groups

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Motivating problem:

Give an example of a torsion-free, finitely generated group G,

and a manifold M, (not S^1 or $\mathbb{R}$)

such that $G \not\subseteq \text{Homeo}(M)$.

Open problem:

Give an example of a torsion-free, finitely generated group G,

and a manifold M, (not S^1 or $\mathbb{R}$)

such that $G \not\subseteq \text{Homeo}(M)$.

In particular,

we know no torsion free f.g. groups that do not act faithfully on Σ_g ... or even D^2.

Known results

Theorem (Witte Morris [Mo11])
$G \subset \mathrm{SL}(n, \mathbb{Z})$ finite index, $n \geq 3$.
Any homomorphism $\phi : G \to \mathrm{Homeo}(S^1)$ has finite image.

- Analogous questions for $\mathrm{Diff}_\mu(M)$ *"Zimmer program"* (see [Fi11])

Theorem (Franks–Handel [FH06])
$G \subset \mathrm{SL}(n, \mathbb{Z})$ finite index, $n \geq 3$. $\Sigma =$ surface.
Any homomorphism $\phi : G \to \mathrm{Diff}_\mu(\Sigma)$ has finite image

F–H main technique: distorted subgroups.

Distortion in finitely generated groups

Definition
$G \subset H$ is *distorted* if $G \hookrightarrow H$ is not a Q.I. embedding.

Special case:
$\langle g \rangle \subset H$ is *distorted* if $\lim_{n\to\infty} \frac{\|g^n\|}{n} = 0$

$\|\cdot\| =$ word norm on H

Distortion in Homeo(Σ) (not finitely generated)

Definition
$G \subset \text{Homeo}(M)$ (or $\text{Diff}(M)$) is *distorted*...
if there exists a finitely generated subgroup $H \subset \text{Homeo}(M)$,
and $G \subset H$ is distorted.

Idea used in [BIP08], [CF06], [Hu15], [Mil14], [Po02],...

Question

Can we make sense of distortion (word norm, large scale geometry) for non finitely-generated groups?

GGT for non f.g. groups

G locally compact, compactly generated ✓

define word norm w.r.t. any compact generating set... ([CH15])

Exercise: $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is Q.I. embedded

G not locally compact ??

Example: word norms on $\mathbb{R}^\infty$

S, U small neighborhoods of id *(generating sets)*
can have $\| \|_S$ not Q.I. to $\| \|_U$

A new framework

Replace compact (generating set) with... "universally bounded"

Definition ([Ro14])

A set $S \subset G$ has *property (OB)* in G if it has finite diameter in any left-invariant metric on G

OB = Orbites Bornées. Equivalent: $G \curvearrowright X$ isometric action $\Rightarrow S \cdot x$ bounded.

Topological groups: require *compatible* left-inv. metric, continuous action

Example: S compact.

A new framework

Replace compact (generating set) with... "universally bounded"

Definition ([Ro14])

A set $S \subset G$ has *property (OB)* in G if it has finite diameter in any left-invariant metric on G

Definition

G is (OB)-generated if $\exists$ generating set S with property (OB)

Exercise: S, U are (OB) generating sets $\Rightarrow \| \, \|_S \sim \| \, \|_U$

...can do GGT!

Nice for topological groups

Assume G separable, metrizable.

E.g. Homeo(M), Diff(M), $\mathbb{R}^\infty$, Banach spaces, Lie groups,...

Proposition ([Ro14])

If G is (OB)-generated by open set U, then:

- *$\exists$ compatible left-invariant metric Q.I. to word metric*
  "compatibility"
- *For any compatible left-invariant d, have $d(x, \mathrm{id}) < K\|x\|_U + C$*
  "maximality"

Proof: First part using Birkhoff–Kakutani metrization, second part exercise.
See [Ro14]

Examples

Groups that are (OB) generated by open sets:

- separable Banach space, $+$
- various automorphism groups,
  e.g. affine isometries of ℓ^p, $Aut(T)$,... [Ro14b]
- Diff_μ with L^p metrics... ([BS13], [BK13]...)

Theorem (M–, Rosendal)

- Homeo(M), *for any compact manifold* M.

Moreover, the large-scale geometry of $\mathrm{Homeo}_0(M)$ *reflects the topology of* M*, and the dynamics of group actions on* M.

Distortion revisited

New (old) definition:

$G \subset \text{Homeo}(M)$ is distorted if $G \hookrightarrow \text{Homeo}(M)$ is not a Q.I. embedding

Proposition

$G \subset \text{Homeo}(M)$ *finitely generated, distorted*

$$\Leftrightarrow \exists \textit{ f.g. } H \textit{ with } G \subset H \textit{ distorted.}$$

... but the distortion function may be different? (open Q.)

Results:
Topology of $M \leftrightarrow$ large scale geometry of $\mathrm{Homeo}_0(M)$

- $\mathrm{Homeo}(S^n) \sim *$ (Proved by Calegari–Freedman, de Cornulier [CF06])
- $M \neq S^1$ and $\pi_1(M)$ infinite $\Rightarrow \mathrm{Homeo}_0(M)$ very big

  contains Q.I. embedded $C([0,1],\mathbb{R})$
- Theorem:
  "Geometry of $\pi_1(M)$ visible in lifts of homeomorphisms to $\widetilde{M}$"

  related to bounded cohomology, Q.I.'s and central extensions

  $1 \to \pi_1(M) \to \text{group of lifts} \to \mathrm{Homeo}_0(M) \to 1$
- Have natural word metric, the *fragmentation norm*

Much unknown: e.g. $\pi_1(M)$ finite $\overset{?}{\Rightarrow} \mathrm{Homeo}_0(M)$ bounded?

Fragmentation

Theorem (Edwards–Kirby)

Given $\{B_1, B_2, ..., B_k\}$ *open cover of* M*. There is a neighborhood* U *of* id *in* $\mathrm{Homeo}(M)$ *such that* $g \in U \Rightarrow g = g_1 \circ ... \circ g_k$.
g_i *pointwise fixes* $M \setminus B_i$.

Definition

The *fragmentation norm* is $\| \ \|_U$ *Well defined up to Q.I.*

Key in proof!

Previous notion (Q.I. equivalent):

$$\|g\| = min\{m \mid g = g_1 \circ ... \circ g_m, \ g_i \text{ fixes } M \setminus B_{k_i}\}$$

Related notion: conjugation-invariant fragmentation norm [BIP08]

Lifting to $\widetilde{M}$

Each g_i from fragmentation has canonical lift to $\widetilde{M}$
Can bound word length in $\mathrm{Homeo}_0(M)$ by looking at $\widetilde{M}$...

A revised question

Question

Give examples of finitely generated groups G that don't $Q.I.$ embed in $\mathrm{Homeo}_0(M)$*.*

Give interesting examples of groups G that do Q.I. embed into $\mathrm{Homeo}_0(M)$*.*

Theorem (evidence of something interesting...)

$G = \mathbb{R} \rtimes \mathbb{Z} \subset \mathrm{Homeo}_0(A)$,
but G has no continuous Q.I. embedding into $\mathrm{Homeo}_0(A)$*.*

More generally...

Problem

Generalize GGT to (non locally-compact) OB–generated groups.

Are there hyperbolic groups?
an interesting theory of ends? growth? ... ??

Some references (not a complete list!)

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