

The Well-Ordering Theorem

one of the Greatest Mathematical Controversies of All Time

Recall that the set of natural numbers with the order $<$ is *well-ordered*. In general, a set (such as \mathbb{N}) with some order ($<$) is called *well-ordered* if any nonempty subset has a least element. The set of even numbers and the set $\{1, 5, 17, 12\}$ with our usual order on numbers are two more examples of well-ordered sets and you can check this. However, the set of integers with our usual ordering on it is *not* well-ordered, neither is the set of rational numbers, nor the set of all *positive* rational numbers.

Whether a set is well-ordered or not depends on how you define the order on the set. Although \mathbb{Z} with our usual $<$ is not *well-ordered*, we can put another funny kind of order on it which does make it well-ordered. Let's call this order \prec instead of $<$. We define it as follows:

$a \prec b$ whenever $|a| < |b|$ or if $|a| = |b|$ and a is a negative number and b is a positive number.

Here, $|a|$ means the *absolute value* of a , this is just a if a is a positive number; if a is negative you forget the minus sign. So $|-3| = 3$. As an example of our funny order, we have:

$2 \prec 7$,
 $-2 \prec 7$,
 $2 \prec -7$,
 $-2 \prec -7$ and
 $-2 \prec 2$.

In summary, you just ignore the minus signs on numbers, unless you're trying to compare a and $-a$, in which case the negative number is smaller.

Which axioms does this order satisfy? If you check, you'll see that O1 (trichotomy) and O2 (transitivity) are both true, but O3 and O4 (addition and multiplication for inequalities) are not. The properties that only have to do with the order are okay, those that use the operations $+$ and \times are not. But many mathematicians are interested in sets, and not-so-interested in the operations of arithmetic, so they only require that orders on sets satisfy O1 and O2. Our order \prec is just fine by their standards. You can also check that \prec makes \mathbb{Z} well ordered. Take any subset of \mathbb{Z} . Does it have an element which is less (according to \prec) than all the others?

Having come up with a well-ordering for \mathbb{Z} we come to the (seemingly naïve!) question that sparked one of the most heated debates in the history of mathematics: Is it true that for *any* set S , there is some order on S that satisfies O1 and O2 and makes S well-ordered? We did it for \mathbb{Z} . If you think very hard, you might be able to come up with an unusual order on \mathbb{Q} that works. But can you imagine such an order on \mathbb{R} ? What would this order look like? What about sets made up of things that are not numbers?

The positive answer to our question – the principle that yes, every single set admits some $<$ satisfying O1 and O2 that makes the set well-ordered – is called the *well-ordering theorem*. For decades, there was a veritable mathematical flame war surrounding whether this theorem was true or not. Logician Georg Cantor, the founder of modern set theory,

started the debate in 1883 by declaring that, in his opinion, the well-ordering theorem was a “fundamental law of thought.” (!) Although this sounds like a radical claim – and it was at the time – Cantor had a pretty good explanation of why he thought it was true. Not too long afterwards, mathematician Ernst Zermelo gave a much clearer exposition of Cantor’s ideas, and eventually went on to show that the well-ordering theorem had many beautiful consequences in mathematics. However, the further these ideas spread, the more mathematicians disagreed with them. Émile Borel – a prominent mathematician at the time – read Zermelo’s work and publicly declared “any argument [along the lines of Zermelo’s] is outside the domain of mathematics.” To which Zermelo responded contrariwise “[the assumption of mine that you question] has already been used, and successfully at that, in the most diverse fields of mathematics.” Moreover, he added, “such an extensive use of a principle can be explained only by its self-evidence.” A fundamental law of thought indeed! Paradoxically, Zermelo was never able to give an explicit *example* of a well-ordering for most sets, including the set of real numbers. He just asserted that it followed from “self-evident principles” that such an order must be possible. The debate continued.

In 1904, Julius König thought he had resolved the question once and for all, and answered it in the negative. In front of an international audience of mathematicians, he presented a *proof* that it was absolutely impossible to put an order on the set \mathbb{R} that made it well-ordered, thus showing the well-ordering principle was false. Unfortunately for him, his proof was soon shown to be fatally flawed and the question still open.

So what do we believe today? First of all, it has been shown that if you want to believe the well-ordering theorem, then it must be taken as an *axiom*. It cannot be proved using the other, commonly accepted axioms of set theory, much like how our cancellation axiom (C) for the integers couldn’t be deduced from our other axioms. But, again like (C) for the integers, you can’t use the axioms that you already have in order to disprove it (so König could never be right!). Also like cancellation, adding the well-ordering theorem to the list of axioms allowed mathematicians to prove some beautiful mathematical theorems that were widely believed to be true.¹ In fact, the modern theory of *infinite numbers*² is based, thanks to work of Zermelo, on the well-ordering theorem. Today there are still a few oddball mathematicians who do not consider well-ordering to be an acceptable axiom, but the vast majority do. And anyone who has studied set theory – especially the history of set theory – agrees at least a little bit with Zermelo: “The well-ordering theorem is the true fundament of the whole theory of number.”³

¹Unfortunately, it also allowed mathematicians to prove some beautiful theorems that most ordinary people would consider to be ridiculous! Moreover, it has also been proven that it is *impossible* to write down an explicit well-ordering for the set of real numbers.

²Yes, infinite numbers. There is much more to this theory than the familiar symbol ∞ , and we’ll get there by the end of this course

³This, along with most of the other quotations from mathematicians given here is taken from the book *Ernst Zermelo: an approach to his life and work* by Ebbinghaus and Peckhaus. Unlike Logicomix, I don’t recommend this one as light reading. Instead, you should check out *Great Feuds in Mathematics* by Hal Hellman. It has a much more readable chapter called ‘Borel vs. Zermelo’ and explains the ‘axiom of choice’ which is the heart of their debate on well-ordering.

Exercises

- (1) Verify that the set $\{x \in \mathbb{N} \mid x \text{ is even}\}$ with our usual $<$ on numbers is well ordered, and so is $\{1, 5, 17, 12\}$.
- (2) Rank these elements of \mathbb{Z} from least to greatest using the order \prec
 $-12, -10, 0, 7, 9, 10, 16$
- (3) Verify that the order \prec on \mathbb{Z} satisfies O1 and O2.
- (4) Check that \mathbb{Z} with \prec is a well-ordered set.
- (5) Can you define an order on the set of all negative integers that makes this set well ordered? (Hint: what if you take our usual order but backwards...)
- (6) Extreme Challenge: Can you define an order on \mathbb{Q} that makes it well ordered?
- (7) Do you think that you believe the well-ordering principle? What does it mean to believe that the set of real numbers has an order on it making it well-ordered, if you also know that it is impossible for anyone to ever write down this order?
(see footnote 1)