# Realizing maps of braid groups by surface diffeomorphisms

Kathryn Mann

Let $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$ denote the group of smooth diffeomorphisms of the 2-dimensional disc that fix a neighborhood of $\partial\mathbb{D}$ and preserve a set $\mathbf{z}_n$ consisting of $n$ points. Let $\mathrm{Diff}_0(\mathbb{D}, \mathbf{z}_n)$ denote the identity component of this group. Then the *mapping class group* $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)/\mathrm{Diff}_0(\mathbb{D}, \mathbf{z}_n)$ is isomorphic to $\mathrm{Br}_n$, the braid group on $n$ strands.

There is a natural "geometric" map $\psi : \mathrm{Br}_{2g+2} \to \mathrm{Mod}_{g,2}$ induced by lifting mapping classes to a double cover $\Sigma_{g,2}$ of the disc $\mathbb{D}$ ramified over the points of $\mathbf{z}_{2g+2}$. One description of this map is as follows: Each $f \in \mathrm{Diff}(\mathbb{D}, \mathbf{z}_{2g+2})$ has a canonical *lift* to a homeomorphism of the cover $\Sigma_{g,2}$; this is the lift that fixes both boundary components pointwise. This gives an injective map $\Psi : \mathrm{Diff}(\mathbb{D}, \mathbf{z}_{2g+2}) \to \mathrm{Homeo}(\Sigma_{g,2}, \partial\Sigma_{g,2})$, and the induced map on the quotient of these groups by their identity components is exactly $\psi$.

Nariman [2] asks if these lifts can be made smooth: *is there a map* $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_{2g+2}) \to \mathrm{Diff}(\Sigma_{g,2}, \partial\Sigma_{g,2})$ *that induces* $\psi$ *on mapping class groups?* Note that the construction above is inherently non-smooth: unless the derivative of $f \in \mathrm{Diff}(\mathbb{D}, \mathbf{z}_{2g+2})$ at each point $z \in \mathbf{z}$ is a scalar, the lift of $f$ to a homeomorphism of the branched cover will not be differentiable at the branch points. Furthermore, there is some (weak) evidence to suggest that no "smoothing" is possible. For instance, Salter–Tshishiku [4] give obstructions to realizing braid groups by diffeomorphisms, so $\psi$ cannot be obtained by a map that factors through $\mathrm{Br}_{2g+2}$. Work of Hurtado [1] also implies that such a map $\psi$ should essentially be continuous, and that its restriction to the subgroup $\mathrm{Diff}_c(\mathbb{D}, \mathbf{z}_{2g+2})$ of diffeomorphisms fixing a neighborhood of $\mathbf{z}$ (which we know to be nontrivial by [4]) must be obtained by embedding copies of covers of the open, punctured disc into $\Sigma_{g,2}$. This suggests, at least vaguely, that $\psi$ would have to be obtained by branching the punctured disc over $\mathbf{z}$, an inherently non-smooth construction.

In [2], Nariman shows – perhaps surprisingly, given the above – that there is no cohomological obstruction to realizing $\psi$ by a map on diffeomorphism groups. Here we confirm Nariman's result and give an alternative proof, via an explicit construction of a realization.

**Theorem 1.1.** There is a continuous homomorphism $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_{2g+2}) \to \mathrm{Diff}(\Sigma_{g,2}, \partial\Sigma_{g,2})$ that induces the geometric homomorphism $\psi : \mathrm{Br}_{2g+2} \to \mathrm{Mod}_{g,2}$ on mapping class groups.

Simpler versions of the constructions used in the proof of Theorem 1.1 can be used to give a smooth version of Thurston's realization of $\mathrm{Br}_3$ by homeomorphisms of the disc. This answers a question asked by B. Tshishiku.

**Theorem 1.2.** There is a homomorphism $\mathrm{Br}_3 \to \mathrm{Diff}(\mathbb{D}, \mathbf{z}_3)$ such its the composition with the quotient map to $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_3)/\mathrm{Diff}_0(\mathbb{D}, \mathbf{z}_3)$ is the identity homomorphism of $\mathrm{Br}_3$.

## 2 Proof of Theorem 1.1

Our strategy is to first build a map $\phi : \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n) \to \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$. This map will have image in a subgroup that acts on a given neighborhood of $\mathbf{z}_n$ by rigid motions of $\mathbb{D}$, will also induce the identity map $\mathrm{Br}_n \to \mathrm{Br}_n$. Building this map is the bulk of the construction. Given such a map $\phi$, the diffeomorphisms in its image can then be lifted to *diffeomorphisms* of a cover branched over $\mathbf{z}_n$ as described above.

We will use the following two familiar constructions in the proof. These are sketched here for the readers convenience.

**Construction 2.1** (Blow-up)**.** Let $P = \{p_1, p_2, ..., p_n\}$ be a finite set of points in a manifold $S$. The *blowup of $S$ at $P$* is a manifold $\hat{S}$ and map $\Phi : \hat{S} \to S$ that is a diffeomorphism away from $\Phi^{-1}(P)$, and such that each $\Phi^{-1}(p_i)$ a sphere of dimension $\dim(M) - 1$. The manifold $\hat{S}$ can be given a smooth structure identifying $\Phi^{-1}(p_i)$ with the unit tangent sphere at $p_i$. If $G$ is a group of smooth diffeomorphisms preserving $P$, there is a natural, injective homomorphism $\iota : G \to \mathrm{Diff}(\hat{S})$ such that $\Phi_* \circ \iota$ is the identity. If $g(p_i) = p_j$, then $\Phi(g)$ restricts to a map from $\Phi^{-1}(p_i)$ to $\Phi^{-1}(p_j)$ agreeing with the induced map on the space of tangent directions.

**Construction 2.2** (Smoothing actions glued on a codimension 1 submanifold)**.** Let $G$ be a group acting by smooth diffeomorphisms on manifolds $S_1$ and $S_2$. Let $X_1$ and $X_2$ be diffeomorphic connected components of $\partial S_1$ and $\partial S_2$ respectively, and let $S$ be the manifold obtained by gluing $S_1$ and $S_2$ by a diffeomorphism $X_1 \to X_2$. If, for each $g \in G$, the action of $g$ on $X_1$ agrees with that on $X_2$ under the identification used in the gluing, then there is an obvious induced action of $G$ on $S$ by homeomorphisms. However, this is *conjugate* to an action by *smooth diffeomorphisms* on $S$. The conjugacy can be obtained by a map $f : S \to S$ which is the identity outside a tubular neighborhood of the glued boundary components, and in the tubular neighborhood (identified with $X \times [-1, 1]$, with the glued boundary components at $X \times \{0\}$) is locally a very strong contraction at 0. Details are worked out in [3] using the local contraction $(x, y) \mapsto (x, e^{\frac{-1}{e^{-1/|y|}}})$.

Now we proceed with the main part of the proof.

**Construction of $\phi : \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n) \to \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$.** First, apply Construction 2.1 to blow up $\mathbb{D}$ at the set $\mathbf{z}_n$. The new surface obtained (call it $D_0$) has $n+1$ boundary components, one corresponding to the original boundary $\partial$ of the disc, and the others corresponding to the blown up points.

Enumerate $\mathbf{z}_n = \{z_1, z_2, ..., z_n\}$ and for $i = 1, 2, ..., n$, let $D_i$ be a blow-up of $\mathbb{D}$ at $\{z_i\}$. Glue each $D_i$ to $D_0$ along the blow-up of $z_i$, using the identity map on the space of tangent directions at $z_i$. The result is an $(n+1)$-holed sphere. Now embed this $(n+1)$-holed sphere into $\mathbb{D}$ with the boundary component $\partial$ mapping to $\partial\mathbb{D}$. The result is pictured in Figure 1; boundary components of the $D_i$ are labeled by their images under the map from the blow-up construction.

Let $C_1, C_2, ...C_n$ be the connected components of the complement of the image of the embedding (shown in white on the figure). We may arrange the embedding so that each $C_i$ is a round disc of radius $\epsilon$, centered around the marked point $z_i$ on the original disc

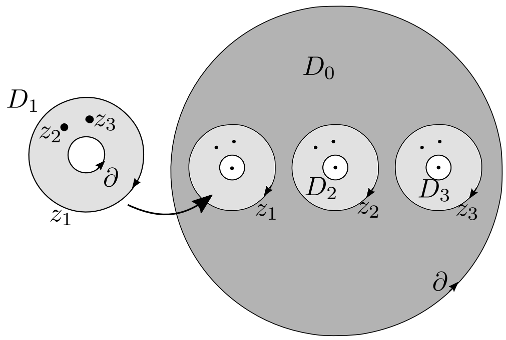


Figure 1: Gluing copies of $\mathbb{D}$ blown up at one point of $\mathbf{z}_3$ into a copy of $\mathbb{D}$ blown up at $\mathbf{z}_3$

$\mathbb{D}$ (shown as midpoints of the white regions of the figure). Constructions 2.1 gives a natural homomorphism from $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$ to $\mathrm{Diff}(D_i)$. These actions of $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$ on the various $D_i$ for $i = 0, 1, ..., n$ agree on their glued boundary components, so Construction 2.2 produces a homomorphism from $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$ to the diffeomorphisms of the $n+1$ holed sphere that was obtained by gluing the $D_i$ together. We identify this surface with the image of its embedding in $\mathbb{D}$. Since elements of $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$ fix a neighborhood of $\partial\mathbb{D}$ pointwise, we may also arrange the embedding so that this action permutes the boundaries of the complementary discs $C_i$ by *rigid translations*. Thus, the action naturally extends to an action on $\mathbb{D}$ by diffeomorphisms, permuting the discs $C_i$ by translations. In particular, the set of midpoints of the $C_i$ is preserved, so this action is by elements of $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$. We let $\phi : \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n) \to \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)$ denote this action.

Finally we check that $\phi$ induces the identity map on the quotient $\mathrm{Br}_n = \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n)/\mathrm{Diff}_0(\mathbb{D}, \mathbf{z}_n)$. By design of our blow-up, gluing, and embedding, if $f(z_i) = z_j$, then $\phi(f)$ maps $D_i$ to $D_j$, hence maps $C_i$ to $C_j$, and its center point $z_i$ to $z_j$.

**Lifting to a branched cover.** Let $n = 2g + 2$. As in the introduction, we have an injective map $\Psi : \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n) \to \mathrm{Homeo}(\Sigma_{g,2}, \partial\Sigma_{g,2})$. Consider the map $\Psi \circ \phi : \mathrm{Diff}(\mathbb{D}, \mathbf{z}_n) \to \mathrm{Homeo}(\Sigma_{g,2}, \partial\Sigma_{g,2})$, which agrees with $\Psi$ on mapping class groups. Each diffeomorphism in the image of $\phi$ has trivial (i.e. constant $\equiv id$) derivative in a neighborhood of each $z \in \mathbf{z}_n$, so its image under $\Psi$ is smooth everywhere. Thus, $\Psi \circ \phi$ gives the desired map $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_n) \to \mathrm{Diff}(\Sigma_{g,2}, \partial\Sigma_{g,2})$. □

# 3 Proof of Theorem 1.2

This section describes a similar blow-up and smoothing trick to turn Thurston's construction from [5] into a realization of $\mathrm{Br}_3$ by diffeomorphisms. Since this construction is unpublished (and relatively quick), we give a self-contained exposition here. Most of the material is well-known.

A standard presentation for $\mathrm{Br}_3$ is $\langle a, b : a^2 = b^3 \rangle$. (To see the relation with mapping classes of $(\mathbb{D}, \mathbf{z}_3)$, take $a$ to be the standard generator supported on a neighborhood of $\{z_1\} \cup \{z_3\}$, swapping these points, and take $b$ the standard generator cyclically permuting the $z_i$.) The group $\mathrm{SL}(2,\mathbb{Z})$ is isomorphic to the quotient of $\mathrm{Br}_3$ by the normal closure of $\{a^4, b^6\}$, taking $a = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The linear action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$ descends to an action by diffeomorphisms on $\mathbb{T}^2$. Taking $[0,1)^2$ as a fundamental domain, this action fixes $(0,0)$ and preserves the set $\{(0,1/2),(1/2,1/2),(1/2,0)\}$. The order two automorphism $x \mapsto -x$ of $\mathbb{R}^2$ defines a quotient map $\pi : \mathbb{T}^2 \to S^2 \cong \mathbb{T}^2/(x = -x)$ that is a degree 2 branched cover, with branching locus the four points $P := \{(0,0),(0,1/2),(1/2,1/2),(1/2,0)\} \subset \mathbb{T}^2$. Considering $S^2$ topologically as $\{(x,y) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} : x + y \leq 1\}$ with appropriate edge identifications, we may also identify $P$ with the set of images of the branch points under $\pi$. The action of $\mathrm{SL}(2,\mathbb{Z})$ descends to an action on $S^2$ by homeomorphisms that are smooth away from $P$. We now modify this using blow-ups to get a smooth action on the disc.

Use Construction 2.1 to blow up $\mathbb{T}^2$ at $P$. The automorphism $x \mapsto -x$ extends to the blow-up, and its quotient under the automorphism is a 4-holed sphere $\hat{S}$, with a map $\Phi : \hat{S} \to S^2$. Away from $P$, $\Phi$ is a diffeomorphism, and for $p \in P$, $\Phi^{-1}(p)$ is naturally identified with the *projectiviezed* tangent space of $\mathbb{T}^2$ at $p$. The action of $\mathrm{SL}(2,\mathbb{Z})$ on the blow-up of $\mathbb{T}^2$ descends naturally to this quotient. It is no longer faithful, but factors through a faithful action of $\mathrm{PSL}(2,\mathbb{Z}) \cong \langle a, b : a^2 = b^3 = 1\rangle$.

Since $(0,0)$ is fixed by $\mathrm{SL}(2,\mathbb{Z})$, one boundary component of $\hat{S}$ is preserved by this action, while the others are permuted transitively. Attach an annulus $S^1 \times [0,1]$ to the preserved boundary component of $\hat{S}$ along $S^1 \times \{0\}$, and embed this new surface into $\mathbb{D}$, with the attached annulus mapping onto a collar neighborhood of $\partial\mathbb{D}$. As in the proof of Theorem 1.1, let $C_1, C_2$ and $C_3$ denote the connected components of the complement of the image of this embedding; with $C_1$ corresponding to the blow-up of $(0,1/2)$, $C_2$ to $(1/2,1/2)$, and $C_3$ to $(1/2,0)$. We may arrange the embedding so that the boundaries $\partial(C_i)$ are round circles of the same radius, and with $z_i \in C_i$ as the center. Since the action on the boundary components of $\hat{S}$ agrees with the action of $\mathrm{SL}(2,\mathbb{Z})$ on the projectivized tangent spaces at points of $P$; we may also arrange this embedding so that, after making suitable identifications $\xi_i : \mathbb{R}P^1 \xrightarrow{\sim} C_i$, such that $\xi_i \xi_j^{-1}$ is a rigid translation, we have that

$$\begin{aligned}
&a|_{C_2} = id \\
&a : C_1 \to C_3 \text{ agrees with } \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R}), \text{ (under our identification)} \\
&b(C_i) = C_{i+1} \text{ (permuting cyclically), and} \\
&b : C_i \to C_{i+1} \text{ agrees with } \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R}).
\end{aligned} \tag{1}$$

We now describe how to extend the action of $\mathrm{SL}(2,\mathbb{Z})$ (which we think of as a non-faithful action of $\mathrm{Br}_3$) on $\hat{S}$, to an action of $\mathrm{Br}_3$ on $\mathbb{D}$ by elements of $\mathrm{Diff}(\mathbb{D}, \mathbf{z}_3)$.

First we extend over the annulus $S^1 \times [0,1]$ which we have embedded as a collar neighborhood of $\partial\mathbb{D}$. Under suitable parameterization, $a$ acts on $S^1 \times \{1\}$ (the preserved boundary component of $\hat{S}$) by a standard order two rotation, and $b$ by an order 3 projectively linear map. Let $b_t$, $0 \leq t \leq 1/2$ be a smooth path of conjugates of $b$ through $\mathrm{PSL}(2,\mathbb{R})$ such that $b_0 = b$, and $b_{1/2} \in \mathrm{SO}(2)$. Now extend this to a smooth path through $\mathrm{SO}(2)$ for $1/2 \leq t \leq 1$, with $b_{1-\epsilon} = \mathrm{id}$ for all small $\epsilon$. Let $a_t$ be a smooth path in $\mathrm{SO}(2)$ from $a_0 = a$ to $a_1 = \mathrm{id}$ such that $a_t^2 = b_t^3$ for all $t$. This gives an extension of the action to a smooth action on the annulus that is identity in a neighborhood of the boundary.

A similar construction allows us to extend over the regions $C_i$. Fix smooth collar neighborhoods of $\partial C_i$ in $C_i$ parametrized by $\partial C_i \times [0,1]$. Applying analogous isotopies to the action of $a$ and $b$ as described in (1) gives an extension of the action to the collar neighborhood such that the action of $a$ and $b$ on $C_i \times \{1\}$ is by rigid translations in $\mathbb{R}^2$. This can then be extended by rigid translations over the complement of these neighborhoods in the $C_i$. The result is an action by diffeomorphisms of $a$ and $b$ on $\mathbb{D}^2$, preserving $\{z_1, z_2, z_3\}$ and such that $a(z_1) = z_3$, and $b(z_i) = z_{i+1}$. Moreover, it is easily verified that the mapping classes of $a$ and $b$ agree with the standard generators of $\mathrm{Br}_3$, as required.

## References


[1] S. Hurtado, *Continuity of discrete homomorphisms of diffeomorphism groups*, Geometry & Topology 19 (2015) 2117–2154.

[2] S. Nariman, *Braid groups and discrete diffeomorphisms of the punctured disc*. Preprint. arXiv:1511.09369 (2016)

[3] K. Parkhe, *Smooth gluing of group actions and applications*. Proc. AMS 143.1 (2015), 203–212.

[4] N. Salter, B. Tshishiku *On the non-realizability of braid groups by diffeomorphisms*, Bull. Lond. Math. Soc. 48 no. 3, (2016), 457–471.

[5] W. Thurston, *Realizing the braid group by homeomorphisms*. `http://mathoverflow.net/questions/55555/realizing-braid-group-by-homeomorphisms`, February 2011.