Realizing maps of braid groups by surface diffeomorphisms

Kathryn Mann

Let $\text{Diff}(\mathbb{D}, z_n)$ denote the group of smooth diffeomorphisms of the 2-dimensional disc that fix a neighborhood of $\partial \mathbb{D}$ and preserve a set $z_n$ consisting of $n$ points. Let $\text{Diff}_0(\mathbb{D}, z_n)$ denote the identity component of this group. Then the mapping class group $\text{Diff}(\mathbb{D}, z_n)/\text{Diff}_0(\mathbb{D}, z_n)$ is isomorphic to $\text{Br}_n$, the braid group on $n$ strands.

There is a natural “geometric” map $\psi: \text{Br}_{2g+2} \to \text{Mod}_{g,2}$ induced by lifting mapping classes to a double cover $\Sigma_{g,2}$ of the disc $\mathbb{D}$ ramified over the points of $z_{2g+2}$. One description of this map is as follows: Each $f \in \text{Diff}(\mathbb{D}, z_{2g+2})$ has a canonical lift to a homeomorphism of the cover $\Sigma_{g,2}$; this is the lift that fixes both boundary components pointwise. This gives an injective map $\Psi: \text{Diff}(\mathbb{D}, z_{2g+2}) \to \text{Homeo}(\Sigma_{g,2}, \partial \Sigma_{g,2})$, and the induced map on the quotient of these groups by their identity components is exactly $\psi$.

Nariman [2] asks if these lifts can be made smooth: is there a map $\text{Diff}(\mathbb{D}, z_{2g+2}) \to \text{Diff}(\Sigma_{g,2}, \partial \Sigma_{g,2})$ that induces $\psi$ on mapping class groups? Note that the construction above is inherently non-smooth: unless the derivative of $f \in \text{Diff}(\mathbb{D}, z_{2g+2})$ at each point $z \in \mathbb{D}$ is a scalar, the lift of $f$ to a homeomorphism of the branched cover will not be differentiable at the branch points. Furthermore, there is some (weak) evidence to suggest that no “smoothing” is possible. For instance, Salter–Tshishiku [4] give obstructions to realizing braid groups by diffeomorphisms, so $\psi$ cannot be obtained by a map that factors through $\text{Br}_{2g+2}$. Work of Hurtado [1] also implies that such a map $\psi$ should essentially be continuous, and that its restriction to the subgroup $\text{Diff}_c(\mathbb{D}, z_{2g+2})$ of diffeomorphisms fixing a neighborhood of $z$ (which we know to be nontrivial by [4]) must be obtained by embedding copies of covers of the open, punctured disc into $\Sigma_{g,2}$. This suggests, at least vaguely, that $\psi$ would have to be obtained by branching the punctured disc over $z$, an inherently non-smooth construction.

In [2], Nariman shows – perhaps surprisingly, given the above – that there is no cohomological obstruction to realizing $\psi$ by a map on diffeomorphism groups. Here we confirm Nariman’s result and give an alternative proof, via an explicit construction of a realization.

**Theorem 1.1.** There is a continuous homomorphism $\text{Diff}(\mathbb{D}, z_{2g+2}) \to \text{Diff}(\Sigma_{g,2}, \partial \Sigma_{g,2})$ that induces the geometric homomorphism $\psi: \text{Br}_{2g+2} \to \text{Mod}_{g,2}$ on mapping class groups.

Simpler versions of the constructions used in the proof of Theorem 1.1 can be used to give a smooth version of Thurston’s realization of $\text{Br}_3$ by homeomorphisms of the disc. This answers a question asked by B. Tshishiku.

**Theorem 1.2.** There is a homomorphism $\text{Br}_3 \to \text{Diff}(\mathbb{D}, z_3)$ such its the composition with the quotient map to $\text{Diff}(\mathbb{D}, z_3)/\text{Diff}_0(\mathbb{D}, z_3)$ is the identity homomorphism of $\text{Br}_3$. 

1
2 Proof of Theorem 1.1

Our strategy is to first build a map \( \phi : \text{Diff}(\mathbb{D}, z_n) \to \text{Diff}(\mathbb{D}, z_n) \). This map will have image in a subgroup that acts on a given neighborhood of \( z_n \) by rigid motions of \( \mathbb{D} \), will also induce the identity map \( \text{Br}_n \to \text{Br}_n \). Building this map is the bulk of the construction. Given such a map \( \phi \), the diffeomorphisms in its image can then be lifted to \textit{diffeomorphisms} of a cover branched over \( z_n \) as described above.

We will use the following two familiar constructions in the proof. These are sketched here for the readers convenience.

**Construction 2.1** (Blow-up). Let \( P = \{p_1, p_2, ..., p_n\} \) be a finite set of points in a manifold \( S \). The \textit{blowup of \( S \) at \( P \)} is a manifold \( \hat{S} \) and map \( \Phi : \hat{S} \to S \) that is a diffeomorphism away from \( \Phi^{-1}(P) \), and such that each \( \Phi^{-1}(p_i) \) is a sphere of dimension \( \dim(M) - 1 \). The manifold \( \hat{S} \) can be given a smooth structure identifying \( \Phi^{-1}(p_i) \) with the unit tangent sphere at \( p_i \). If \( G \) is a group of smooth diffeomorphisms preserving \( P \), there is a natural, injective homomorphism \( \iota : G \to \text{Diff}(\hat{S}) \) such that \( \Phi \circ \iota \) is the identity. If \( g(p_i) = p_j \), then \( \Phi(g) \) restricts to a map from \( \Phi^{-1}(p_i) \) to \( \Phi^{-1}(p_j) \) agreeing with the induced map on the space of tangent directions.

**Construction 2.2** (Smoothing actions glued on a codimension 1 submanifold). Let \( G \) be a group acting by smooth diffeomorphisms on manifolds \( S_1 \) and \( S_2 \). Let \( X_1 \) and \( X_2 \) be diffeomorphic connected components of \( \partial S_1 \) and \( \partial S_2 \) respectively, and let \( S \) be the manifold obtained by gluing \( S_1 \) and \( S_2 \) by a diffeomorphism \( X_1 \to X_2 \). If, for each \( g \in G \), the action of \( g \) on \( X_1 \) agrees with that on \( X_2 \) under the identification used in the gluing, then there is an obvious induced action of \( G \) on \( S \) by homeomorphisms. However, this is \textit{conjugate} to an action by \textit{smooth diffeomorphisms} on \( S \). The conjugacy can be obtained by a map \( f : S \to S \) which is the identity outside a tubular neighborhood of the glued boundary components, and in the tubular neighborhood (identified with \( X \times [-1, 1] \), with the glued boundary components at \( X \times \{0\} \)) is locally a very strong contraction at \( 0 \).

Details are worked out in [3] using the local contraction \( (x, y) \mapsto (x, e^{-\frac{1}{1+y}}) \).

Now we proceed with the main part of the proof.

**Construction of \( \phi : \text{Diff}(\mathbb{D}, z_n) \to \text{Diff}(\mathbb{D}, z_n) \).** First, apply Construction 2.1 to blow up \( \mathbb{D} \) at the set \( z_n \). The new surface obtained (call it \( D_0 \)) has \( n + 1 \) boundary components, one corresponding to the original boundary \( \partial \) of the disc, and the others corresponding to the blown up points.

Enumerate \( z_n = \{z_1, z_2, ..., z_n\} \) and for \( i = 1, 2, ..., n \), let \( D_i \) be a blow-up of \( \mathbb{D} \) at \( \{z_i\} \). Glue each \( D_i \) to \( D_0 \) along the blow-up of \( z_i \), using the identity map on the space of tangent directions at \( z_i \). The result is an \( (n + 1) \)-holed sphere. Now embed this \( (n + 1) \)-holed sphere into \( \mathbb{D} \) with the boundary component \( \partial \) mapping to \( \partial \mathbb{D} \). The result is pictured in Figure 1; boundary components of the \( D_i \) are labeled by their images under the map from the blow-up construction.

Let \( C_1, C_2, ..., C_n \) be the connected components of the complement of the image of the embedding (shown in white on the figure). We may arrange the embedding so that each \( C_i \) is a round disc of radius \( \epsilon \), centered around the marked point \( z_i \) on the original disc.
Figure 1: Gluing copies of $D$ blown up at one point of $z_3$ into a copy of $D$ blown up at $z_3$

$D$ (shown as midpoints of the white regions of the figure). Constructions 2.1 gives a natural homomorphism from $\text{Diff}(D, z_n)$ to $\text{Diff}(D_i)$. These actions of $\text{Diff}(D, z_n)$ on the various $D_i$ for $i = 0, 1, ..., n$ agree on their glued boundary components, so Construction 2.2 produces a homomorphism from $\text{Diff}(D, z_n)$ to the diffeomorphisms of the $n + 1$ holed sphere that was obtained by gluing the $D_i$ together. We identify this surface with the image of its embedding in $D$. Since elements of $\text{Diff}(D, z_n)$ fix a neighborhood of $\partial D$ pointwise, we may also arrange the embedding so that this action permutes the boundaries of the complementary discs $C_i$ by rigid translations. Thus, the action naturally extends to an action on $D$ by diffeomorphisms, permuting the discs $C_i$ by translations. In particular, the set of midpoints of the $C_i$ is preserved, so this action is by elements of $\text{Diff}(D, z_n)$. We let $\phi : \text{Diff}(D, z_n) \to \text{Diff}(D, z_n)$ denote this action.

Finally we check that $\phi$ induces the identity map on the quotient $\text{Br}_n = \text{Diff}(D, z_n)/\text{Diff}_0(D, z_n)$.

Lifting to a branched cover. Let $n = 2g + 2$. As in the introduction, we have an injective map $\Psi : \text{Diff}(D, z_n) \to \text{Homeo}(\Sigma_{g,2}, \partial \Sigma_{g,2})$. Consider the map $\Psi \circ \phi : \text{Diff}(D, z_n) \to \text{Homeo}(\Sigma_{g,2}, \partial \Sigma_{g,2})$, which agrees with $\Psi$ on mapping class groups. Each diffeomorphism in the image of $\phi$ has trivial (i.e. constant $\equiv \text{id}$) derivative in a neighborhood of each $z \in z_n$, so its image under $\Psi$ is smooth everywhere. Thus, $\Psi \circ \phi$ gives the desired map $\text{Diff}(D, z_n) \to \text{Diff}(\Sigma_{g,2}, \partial \Sigma_{g,2})$.

3 Proof of Theorem 1.2

This section describes a similar blow-up and smoothing trick to turn Thurston’s construction from [5] into a realization of $\text{Br}_3$ by diffeomorphisms. Since this construction is unpublished (and relatively quick), we give a self-contained exposition here. Most of the material is well-known.

A standard presentation for $\text{Br}_3$ is $\langle a, b : a^2 = b^3 \rangle$. (To see the relation with mapping classes of $(D, z_3)$, take $a$ to be the standard generator supported on a neighborhood of $\{z_1\} \cup \{z_3\}$, swapping these points, and take $b$ the standard generator cyclically permuting
the $z_i$.) The group $\text{SL}(2, \mathbb{Z})$ is isomorphic to the quotient of $\text{Br}_3$ by the normal closure of $\{a^2, b^0\}$, taking $a = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ and $b = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right)$.

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The linear action of $\text{SL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ descends to an action by diffeomorphisms on $T^2$. Taking $[0, 1)^2$ as a fundamental domain, this action fixes $(0, 0)$ and preserves the set $\{(0, 1/2), (1/2, 1/2), (1/2, 0)\}$. The order two automorphism $x \mapsto -x$ of $\mathbb{R}^2$ defines a quotient map $\pi : T^2 \to S^2 \cong T^2/(x = -x)$ that is a degree 2 branched cover, with branching loci the four points $P := \{(0, 0), (0, 1/2), (1/2, 1/2), (1/2, 0)\} \subset T^2$.

Considering $S^2$ topologically as $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x + y \leq 1\}$ with appropriate edge identifications, we may also identify $P$ with the set of images of the branch points under $\pi$. The action of $\text{SL}(2, \mathbb{Z})$ descends to an action on $S^2$ by homeomorphisms that are smooth away from $P$. We now modify this using blow-ups to get a smooth action on the disc.

Use Construction 2.1 to blow up $T^2$ at $P$. The automorphism $x \mapsto -x$ extends to the blow-up, and its quotient under the automorphism is a 4-holed sphere $\hat{S}$, with a map $\Phi : \hat{S} \to S^2$. Away from $P$, $\Phi$ is a diffeomorphism, and for $p \in P$, $\Phi^{-1}(p)$ is naturally identified with the projectivized tangent space of $T^2$ at $p$. The action of $\text{SL}(2, \mathbb{Z})$ on the blow-up of $T^2$ descends naturally to this quotient. It is no longer faithful, but factors through a faithful action of $\text{PSL}(2, \mathbb{Z}) \cong \langle a, b : a^2 = b^3 = 1 \rangle$.

Since $(0, 0)$ is fixed by $\text{SL}(2, \mathbb{Z})$, one boundary component of $\hat{S}$ is preserved by this action, while the others are permuted transitively. Attach an annulus $S^1 \times [0, 1]$ to the preserved boundary component of $\hat{S}$ along $S^1 \times \{0\}$, and embed this new surface into $\mathbb{D}$, with the attached annulus mapping onto a collar neighborhood of $\partial \mathbb{D}$. As in the proof of Theorem 1.1, let $C_1, C_2$ and $C_3$ denote the connected components of the complement of the image of this embedding; with $C_1$ corresponding to the blow-up of $(0, 1/2)$, $C_2$ to $(1/2, 1/2)$, and $C_3$ to $(1/2, 0)$. We may arrange the embedding so that the boundaries $\partial(C_i)$ are round circles of the same radius, and with $z_i \in C_i$ as the center. Since the action on the boundary components of $\hat{S}$ agrees with the action of $\text{SL}(2, \mathbb{Z})$ on the projectivized tangent spaces at points of $P$; we may also arrange this embedding so that, after making suitable identifications $\xi_i : \mathbb{R}P^1 \sim \sim C_i$, such that $\xi_i^{-1}$ is a rigid translation, we have that

$$a|_{C_2} = \text{id}$$

$$a : C_1 \to C_3 \text{ agrees with } \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \in \text{PSL}(2, \mathbb{R}), \text{ (under our identification)}$$

$$b(C_i) = C_{i+1} \text{ (permuting cyclically), and}$$

$$b : C_i \to C_{i+1} \text{ agrees with } \left( \begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right) \in \text{PSL}(2, \mathbb{R}).$$

We now describe how to extend the action of $\text{SL}(2, \mathbb{Z})$ (which we think of as a non-faithful action of $\text{Br}_3$) on $\hat{S}$, to an action of $\text{Br}_3$ on $\mathbb{D}$ by elements of $\text{Diff}(\mathbb{D}, z_3)$.

First we extend over the annulus $S^1 \times [0, 1]$ which we have embedded as a collar neighborhood of $\partial \mathbb{D}$. Under suitable parameterization, $a$ acts on $S^1 \times \{1\}$ (the preserved boundary component of $\hat{S}$) by a standard order two rotation, and $b$ by an order 3 projectively linear map. Let $b_t, 0 \leq t \leq 1/2$ be a smooth path of conjugates of $b$ through $\text{PSL}(2, \mathbb{R})$ such that $b_0 = b$, and $b_{1/2} \in \text{SO}(2)$. Now extend this to a smooth path through $\text{SO}(2)$ for $1/2 \leq t \leq 1$, with $b_{1/2}^t = \text{id}$ for all small $\epsilon$. Let $a_t$ be a smooth path in $\text{SO}(2)$ from $a_0 = a$ to $a_1 = \text{id}$ such that $a_t^2 = b_t^3$ for all $t$. This gives an extension of the action to a smooth action on the annulus that is identity in a neighborhood of the boundary.
A similar construction allows us to extend over the regions $C_i$. Fix smooth collar neighborhoods of $\partial C_i$ in $C_i$ parametrized by $\partial C_i \times [0,1]$. Applying analogous isotopies to the action of $a$ and $b$ as described in (1) gives an extension of the action to the collar neighborhood such that the action of $a$ and $b$ on $C_i \times \{1\}$ is by rigid translations in $\mathbb{R}^2$. This can then be extended by rigid translations over the complement of these neighborhoods in the $C_i$. The result is an action by diffeomorphisms of $a$ and $b$ on $\mathbb{D}^2$, preserving $\{z_1, z_2, z_3\}$ and such that $a(z_1) = z_3$, and $b(z_i) = z_{i+1}$. Moreover, it is easily verified that the mapping classes of $a$ and $b$ agree with the standard generators of $\text{Br}_3$, as required.

References


