Math 113 Homework 7. Due 10/27

Reading corresponding to lectures:
Tuesday 10/20 DF 7.1,
Thursday 10/22 DF 7.2
Tuesday 10/27 DF 7.3, 7.4

Problems to hand in:

1. Do the following problems from DF section 7.1: 2, 3, 5, 6, 11, 15, 20.
2. Let R be a ring, and a and $b \in \mathbb{R}$. Prove that $(-a)(-b) = ab$.
3. Let R be a ring with 1. Prove that the units of R form a group (with binary operation $\times$).
4. Recall the *norm* of a complex number $a + bi$ is defined to be $\sqrt{a^2 + b^2}$. It is the length of the vector in the complex plane that represents $a + bi$. The purpose of this problem is to generalize this idea to the quadratic field $\mathbb{Q}(\sqrt{D})$. (see DF page 229-230)

   For $\alpha = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{D})$, define $N(a + b\sqrt{D}) = a^2 - b^2 D$.

   (a) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in Q(\sqrt{D})$
   (b) Assume that $D \equiv 2 \mod 4$ or $D \equiv 3 \mod 4$. Show that $N(\alpha) \in \mathbb{Z}$ for every $\alpha \in \mathbb{Z}[\sqrt{D}]$. (note: you don't *need* to use the fact that $D \equiv 2$ or $D \equiv 3 \mod 4$...)
   (c) Assume that $D \equiv 1 \mod 4$. Show that $N(\alpha) \in \mathbb{Z}$ for every $\alpha \in \{a + b(\frac{1+\sqrt{D}}{2}) \mid a, b \in \mathbb{Z}\}$
   (d) Prove that α is a unit in $\mathbb{Z}[\sqrt{D}]$ ($D \equiv 2, 3 \bmod 4$) or $\{a + b(\frac{1+\sqrt{D}}{2}) \mid a, b \in \mathbb{Z}\}$($D \equiv 1 \bmod 4$ if and only if $N(\alpha) = \pm 1$.

   Not to be graded: What are the units in $\mathbb{C}$? What are the units in $\mathbb{Q}(\sqrt{-1})$? What are the units in $\mathbb{Z}[\sqrt{-1}]$?
5. How many elements are in the matrix ring $M_n(R)$ when $R = \mathbb{Z}/m\mathbb{Z}$? .
6. Do the following problems from DF section 7.2: 1 part c) only, 3 part b) only (you may assume that $R[[x]]$ is a commutative ring with 1), 8.