Math 113 Homework 6. Due 10/20

Reading corresponding to lectures:
- Thursday, 10/16: DF 4.5, or notes on Sylow's theorem, on website. DF Section 4.4
- Tuesday 10/16 DF 7.1

Problems to hand in:

1. Let p and q be different primes, and let G be the product group $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/q\mathbb{Z})$. What is the order of G? Use Sylow's theorem to prove that G is cyclic.
   (Hint: In an abelian group, all subgroups are normal. How many Sylow p- and Sylow q- subgroups can there be?)
   (Note: it is also possible to give a generator for G, but I am asking you to use Sylow's theorem instead.)

2. As a partial converse to the first question, suppose now that m and n have a common factor other than 1. Prove that $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ is *not* cyclic.

3. For each prime p dividing 50, describe all of the Sylow p-subgorups of D_{50}.
   (hint: this looks like something on the midterm...)

4. Describe all the Sylow 3-subgroups of S_5

5. Do the following problems from DF Section 4.4: 1, 3.

6. **[OPTIONAL, not to hand in:]** Classification of groups of order 21
   For those of you who are ambitions, here's an outline of how to classify the groups of order 21 (there are exactly two isomorphism classes).
   (a) Show that, either G is abelian, or $n_7 = 1$ and $n_3 = 7$.
   (b) Now assume that $n_7 = 1$ and $n_3 = 7$. Let x be a generator for the Sylow 7-subgroup. Let y be a generator for some Sylow 3-subgroup. Show that x and y generate G. (hint: look at cosets of the Sylow 7-subgroup).
   (c) Since the Sylow 7-subgroup $\langle x \rangle$ is normal, $yxy^{-1} = x^k$ for some k. Show that
   $$k^3 \equiv 1 (\text{mod } 7)$$
   (d) If $k = 2$, then G is generated by x and y with relations $x^7 = e$, $y^3 = e$, $yxy^{-1} = x^2$. Show that, in a group with these generators and relations, every element can be written in the form $x^k y^j$, with $0 \leq k \leq 6$ and $0 \leq y \leq 2$. In particular, the group has 21 elements.
   (e) If $k = 4$, then let $z = y^2$, and show that $zxz^{-1} = x^2$. (thus, we are in the case above).