Math 113 Homework 5. Due 10/6

Reading corresponding to lectures:
- Tuesday 9/29: DF 3.1, 3.2
- Thursday 10/1 DF 3.2, 3.3. You are not responsible for knowing the material in 3.4, but we will talk about it. We won't cover 3.5 – you did the most important part on problem set 4!

Problems to hand in:

1. (The relation "is a normal subgroup of" is not transitive). Let $G = D_8$. Show that $\langle s, r^2 \rangle$ is a normal subgroup of D_8 and that $\langle s \rangle$ is a normal subgroup of $\langle s, r^2 \rangle$, but $\langle s \rangle$ is *not* a normal subgroup of D_8.

2. DF section 3.1: 14 parts a and b only, 16, 20.

3. Do the following problems from DF section 3.2: 5, 11, 12. For problem 11, you may assume that $|G : H|$ and $|H : K|$ are finite, but not that G is finite!

4. Let H and K be subgroups of G, with $H \neq K$.
   (a) Prove that if $|H|$ and $|K|$ are relatively prime, then $H \cap K = \{e\}$.
   (b) Prove that if $|H| = |K| = p$, where p is prime, then $H \cap K = \{e\}$.
   (c) Suppose G is a group with $|G| = 35$. Show that G has at most 8 subgroups of order 5, and at most 5 subgroups of order 7. (hint: use part b)
   (d) Using the above, show that if $|G| = 35$, then G contains an element of order 5. (hint: if G is cyclic, then $G = Z_{35}$ which has an element of order 5. If G is not cyclic, what are the possible orders of elements? Why can't every element have order 7?)
   Now show using the same kind of argument that G has an element of order 7.

   Remark: This is not just a special fact about groups of order 35 – Cauchy's theorem (Theorem 11 in DF 3.2) says that, for each prime dividing the order of a group G, there is an element in G of that order.

5. (Orbits and cosets) Let G act on a set A. For each $a \in A$, let $\text{stab}(a) = \{g \in G \mid g{\cdot}a = a\}$. We showed earlier that $\text{stab}(a)$ was a subgroup.
   (a) Let $a \in A$ and $x \in G$. Show that the set of left cosets of $\text{stab}(a)$ are in one-to-one correspondence with elements of the orbit of a. (show that $\psi : x\,\text{stab}(a) \mapsto x \cdot a$ is well defined and a bijection).
   (b) (not to hand in:) use this to compare the two proofs we gave of Lagrange's theorem

   The next two questions use material that will be discussed on Thursday 10/1:

6. Use Fermat's little theorem to prove the following: Let $n_1, n_2, ... n_{30}$ be integers, and assume that at least one of them is not divisible by 31.
   Show that $(n_1)^{30} + (n_2)^{30} + ... + (n_{30})^{30}$ is also not divisible by 31.

7. Let $G = GL_2(\mathbb{C})$. You may take it as a fact that $Z(G) = \{\lambda I \mid \lambda \in \mathbb{C} - \{0\}\}$, the multiples of the identity matrix.
   (a) Let $A = SL_2(\mathbb{C})$ be the set of matrices with determinant 1. Let ϕ be the determinant homomorphism, $\phi : GL_2(\mathbb{C}) \to \mathbb{C}^\times$. Use the first isomorphism theorem to describe G/A.
   (b) Now let $B = Z(G)$. Show that $G = AB$.
   (c) What is $A \cap B$?
   (d) What can you conclude using the second isomorphism theorem (what group is isomorphic to G/B?)

   Remark: This group is called the *projective general linear group*, written $PGL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ (the fact that it has two names comes from the isomorphism above!)