Math 113 Homework 4. Due 9/29

Reading corresponding to lectures:
Tuesday 9/22: DF 1.7, (and a small part of 4.2)
Thursday 9/23 DF 3.1 *Read it very carefully!*
Tuesday 9/32 DF 3.1, 3.2

Problems to hand in:

1. Normal subgroups:
   (a) (Refer to the definition of *normal* from problem set 3, and the definition of coset from lecture)
   Let G be a group, and H a subgroup of G. Prove that $gH = Hg$ for all $g \in G$ if and only if H is normal.
   (b) Let A and B be groups. Show that the set $\{(a, 1_B) \mid a \in A\}$ is a normal subgroup of $A \times B$. (hint for a possible approach: is it the kernel of a homomorphism?)

2. Fibers of homomorphisms (to be done after Thursday's lecture). Do these problems from DF Section 3.1: 8, 9, 12

The remaining problems are meant to introduce you to an important concept: the *sign* of a permutation. You will use familiar tools (from chapters 1 and 2): actions, homomorphisms, the symmetric group, matrices...

3. Define an action of S_n on $\mathbb{R}^n$ by
$$\sigma \cdot (x_1, x_2, \dots x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots x_{\sigma^{-1}(n)})$$
   (a) If $\sigma = (1234)$ in S^4, what is $\sigma(x_1, x_2, x_3, x_4)$? In general (for any permutation σ), in what new position does the action of σ put x_k?
   (b) Prove that this satisfies the two axioms in the definition of *action*.

4. (a) Let $(\mathbb{R})_n LG$ denote the set of $n \times n$ invertible matrices with real entries, with a binary operation $*$ defined by
$$A * B = BA$$
   Here the right hand side is the usual matrix multiplication.
   Prove that $(\mathbb{R})_n LG$ is a group [1]
   (b) Define a function $\phi : S_3 \to (\mathbb{R})_3 LG$ by:
   $\phi(e) = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ $\phi((12)) = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$ $\phi((23)) = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$
   $\phi((13)) = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$ $\phi((123)) = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$ $\phi((132)) = \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$
   Show that ϕ is a homomorphism. (you may either check this directly, or explain using facts about elementary matrices)

[1] You may have noticed that the name for this group is $GL_n(\mathbb{R})$ backwards. The group is $GL_n(\mathbb{R})$ with a "backwards" version of matrix multiplication! The reason for doing this instead of the usual $GL_n(\mathbb{R})$ has to do with the fact that function composition is a "backwards" kind of multiplication and how our action works – analogous to how we read braids from bottom to top when we thought of them as permutations. You can read about the problem here: `http://en.wikipedia.org/wiki/Permutation_matrix`

5. A *permutation matrix* is a $n \times n$ matrix, where each entry is either 0 or 1, and the number 1 appears exactly once in each column and once in each row. For example, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is a 3×3 permutation matrix but $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is not.

   (a) How many $n \times n$ permutation matrices are there? (give a formula involving n and explain)

   (b) Show that ϕ from the question above is an isomorphism onto the set of 3×3 permutation matrices inside of $(\mathbb{R})_3 LG$. (and hence the permutation matrices form a group).

   (c) Show that, for each transposition σ in S_3 the action of σ on $\mathbb{R}^3$ specified in question 3 agrees with the action defined by

   $$\sigma \cdot (x_1, x_2, x_3) = \phi(\sigma) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

   (on the right hand side, you are multiplying matrices and vectors in the usual way)

   [optional bonus: use the properties of actions and homomorphisms and the fact that transpositions generate S^3 to prove that this holds not just for transpositions, but for all elements of S^3].

6. By a similar procedure, we can define a homomorphism $\phi : S_n \to (\mathbb{R})_n LG$ by specifying $\phi(\sigma)$ to be the matrix that maps $(x_1, x_2, \ldots x_n)$ to $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots x_{\sigma(n)})$.
   [you do not need to prove that this is a homomorphism, but you should think about why it is]

   (a) Describe, in words, the matrices that correspond to *transpositions* (the definition of transposition is on a previous homework).

   (b) Show that a matrix corresponding to a transposition always has determinant -1. (you may use standard facts about elementary matrices to make this very easy. Or you can prove it yourself)

   (c) Deduce that an element $\sigma \in S_n$ can be written as a product of an even number of transpositions if and only if $\det(\phi(\sigma)) = 1$.

Important note: $\det(\phi(\sigma))$ is called the *sign* of the permutation σ.
The map $\epsilon : S_n \to \mathbb{R}$, defined by $\epsilon(\sigma) = det(\phi(\sigma))$ is a homomorphism from S_n to the cyclic group with two elements, here thought of as $\{-1, +1\}$ with multiplication. The kernel of this homomorphism is called the *alternating group* A_n.

Astounding fact! A single permutation can be written as a product of transpositions in many ways. For example, $(123) = (13)(12) = (12)(13)(12)(13)$. What you just showed is that the number of transpositions is either always even or always odd!

This material is presented in Section 3.5 of DF, although with a slightly different approach.