Math 113 Homework 3. Due 9/22

Reading corresponding to lectures:
Tuesday 9/15: DF 2.1, 2.3, some of 2.2
Thursday 9/18 DF 1.7 and 4.1
Tuesday 9/22 – you have a 30 minute test; after which we'll talk more about 4.1

Problems to hand in:

1. Let G be a group and A a subset of G. The *normalizer* of A in G, written $N_G(A)$, is
$$\{g \in G \mid gAg^{-1} = A\}$$
Here, gAg^{-1} means the set $\{gag^{-1} \mid a \in A\}$. (see page 50 of DF)
   (a) Let $G = D_{10}$ and $A = \{r, s\}$. What is rAr^{-1}?
   (b) Let $G = D_{10}$ and $A = \{r, s\}$ List all the elements of $N_G(A)$

2. Do these problems from DF Section 2.3: 2, 3, 11.

3. Let G and H be groups. The *product group* $G \times H$ is the set $\{(g, h) : g \in G, h \in H\}$ with multiplication $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$.
   (a) Let $G = (\mathbb{R} \setminus \{0\}, \times)$ and $H = (\mathbb{R}, +)$ What is the identity element in $G \times H$?
   (b) Prove that $\mathbb{Z} \times \mathbb{Z}$ is not cyclic
   (c) Prove that $\mathbb{Z} \times \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$.

4. A subgroup H of a group G is called *normal* if $N_G(H) = G$
   (a) Let G be an abelian group. Prove that every subgroup of G is normal.
   (b) Give an example of a subgroup of S_4 that is not normal.
   (c) Let $SL_n(\mathbb{R})$ be the group of $n \times n$ invertible matrices with determinant equal to one. Prove that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$. (you need to prove both that it is a subgroup, and that it is normal)
   (d) Let G and H be groups, and let $\phi : G \to H$ be a homomorphism. The kernel of ϕ is $\{g \in G : \phi(g) = e\}$ Prove that the kernel of ϕ is a normal subgroup of G.
   (e) Give a different proof that $SL_n(\mathbb{R})$ is normal in $GL_n(\mathbb{R})$, by showing that determinant is a homomorphism (a homomorphism from what to what?) and using the problem above.

5. Do these problems from DF Section 1.7: 3, 14, 15.

6. Recall the affine group defined in class. $\text{Aff}(\mathbb{R}^2)$ is the set of pairs $(M, \vec{v}) \in GL_2(\mathbb{R}) \times \mathbb{R}^2$ with multiplication $(M, \vec{v})(N, \vec{w}) = (MN, M\vec{w} + \vec{v})$. $\text{Aff}(\mathbb{R}^2)$ acts on vectors in $\mathbb{R}^2$ by $(M, \vec{v}) \cdot \vec{x} = M\vec{x} + \vec{v}$.
   (a) Prove that this defines a group action. (you may follow the argument from class if you like. You do *not* need to prove that $\text{Aff}(\mathbb{R}^2)$ is a group)
   (b) Compute the stabilizer of the vector $(1, 0)$. Is it easy to see that this is a subgroup of $\text{Aff}(\mathbb{R}^2)$?