Math 113 final homework set. Due 12/4

Algebra has a very great and elevating effect, compelling the soul to reason about abstract number... You know how steadily the masters of the art repel and ridicule any one who attempts to divide absolute unity when he is calculating, and if you divide, they multiply, taking care that one shall continue and not become lost in fractions

- from Plato's *Republic*

I hope you have learned to enjoy abstract numbers and not become lost in fields of fractions.

Reading corresponding to lectures:

Tuesday 11/ 25 DF 9.3 (just Gauss' lemma at the beginning), 9.4

Tuesday 12/2 DF Parts of 13.1 – definitions of Field extension, degree, and theorem 4. Also, theorem 14 from 13.2

Thursday 12/ 4 DF 13.3

Problems to hand in:

1. Do the following problems from DF section 9.2: #10
2. Do the following problems from DF section 9.4: #1, 2 a-c, 3.
3. Let $I \subset \mathbb{R}[x]$ be the ideal generated by x^2+1. Prove that $\mathbb{R}[x]/I$ is isomorphic to $\mathbb{C}$ by looking at the "evaluation at i" homomorphism, $\phi : \mathbb{R}[x] \to \mathbb{C}$ defined by $\phi(p(x))p(i)$. What is the kernel?
4. Let F be a field, $a \in F$, and let k be an odd integer. Show that $x^k - a^k$ is reducible in $F[x]$
5. Let $a \in \mathbb{R}$, and let $f(x) \in \mathbb{R}[x]$. Show that the remainder when $f(x)$ is divided by $(x-a)^2$ is equal to
$$f'(a)(x-a) + f(a)$$
Hints: the hard way to do this is to write $f(x) = a_n x^n + ... + a_0$, figure out what its derivative is, do the division algorithm, etc...
The easy way is to say, by the division algorithm $f(x) = q(x)(x-a)^2 + r(x)$ for some $q(x)$ and $r(x)$, take the derivative of both sides of the equation, evaluate at a...
6. (a) Show that $\mathbb{Q}(\sqrt{2})$ is a *vector space* over the field $\mathbb{Q}$, with basis $\{1, \sqrt{2}\}$.
   (b) Is $\{1, 3+5\sqrt{2}\}$ also a basis? What about $\{3, \sqrt{2}, 5\}$?
7. Prove that $\{a + b\sqrt{2} + c\sqrt{3} \mid a, b, c \in \mathbb{Q}\}$ is not a field. Is it even a ring?
(optional bonus: what is the smallest subfield of $\mathbb{R}$ that contains this set?)
8. Extra challenge, not to hand in: *prove a better version of Eisenstein's criterion – do problem in DF 9.4 #17. Start off by imitating our proof... how far does this get you?*