Math 113 homework 0. Due 9/1

This homework does not count towards your grade, however I will read it. It will also help you understand the material covered in class.

Reading:

- Dummit and Foote (henceforth called DF) sections 0.3 and 1.1
- Read the course syllabus and the information on the course webpage.

Problems to hand in:

1. I'd like to get to know you
   (a) What is your major?
   (b) What math courses did you take during the past year? What other math courses are you taking now?
   (c) (optional) Thinking about the math that you've studied in college, what concept or topic did you find most difficult? (also optional: what made it difficult?)

2. Decide whether each of the following is a *binary operation* on the specified set. If it is not, explain why.
   - Composition, on the set of functions $\mathbb{R} \to \mathbb{R}$.
     reminder: for the functions $f(x) = \sin(x)$ and $g(x) = x^2$, their composition is $f \circ g(x) = \sin(x^2)$.
   - Composition, on the set of functions from $\mathbb{R}$ to $\mathbb{R}^2$
   - Taking the average of two numbers, on the set of rational numbers.
   - Taking the maximum of a pair, on the set of real numbers.
   - The operation $\star$ defined on the set of all possible things you can drink by $X \star Y =$ (3 oz of X mixed with 5 oz of Y).
     For example, Gin $\star$ Tonic = a standard gin and tonic.

3. Give an example of another set with a binary operation (not listed above or discussed in class).

4. Which of the binary operations in questions (2) and (3) are associative? Commutative?

5. Is the set of all possible things you can drink, with the operation $\star$ a *group*? Why or why not?

6. Modular arithmetic (residue classes)
   In class, we discussed how $\mathbb{Z}/12\mathbb{Z}$, and more generally $\mathbb{Z}/n\mathbb{Z}$, is a group with the operation of addition, but you can also multiply elements of $\mathbb{Z}/n\mathbb{Z}$. **Read section 0.3 of DF for more details**.
   (a) Prove that $1243 + 1985^2 - 4827 + 4$ does not equal $4839 + 753^3 - 56(81)$ by finding some n so that

$$1243 + 1985^2 - 4827 + 4 \not\equiv 4839 + 753^3 - 56(81) \pmod{n}$$

(b) Let $\mathbb{Z}/n\mathbb{Z}^\times = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid \text{there exists } \bar{c} \in \mathbb{Z}/n\mathbb{Z} \text{ with } \bar{a}\bar{c} = \bar{1}\}$. Show that $(\mathbb{Z}/n\mathbb{Z})^\times$ with multiplication is a group.

(c) (extra bonus) Prove that if a and n have a common factor (or "common divisor") other than 1, then $\bar{a}$ is not an element of $(\mathbb{Z}/n\mathbb{Z})^\times$.
(hint: find b so that $ab = 0 \pmod{n}$. Why does this show that $\bar{a} \notin \mathbb{Z}/n\mathbb{Z}^\times$?)

Remark: in fact, it is true that $\bar{a} \in (\mathbb{Z}/n\mathbb{Z})^\times$ if and only if a and n have no common factors. See exercise 13 in DF section 0.3 for a hint if you're interested. But this is not homework.